CONTROL SYSTEMS THEORY
AND THE
MATHEMATICAL MODELLING AND DESIGN
OF STRUCTURES

A thesis submitted in partial fulfillment of
the requirements for the Degree of
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D. Carmichael

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ABSTRACT

The equations and principles of structural mechanics are discussed within the framework of an hierarchical multilevel systems theory after Clyde. A single-level model is advanced and general results obtained for the single-level, single-goal design problem. The systems viewpoint is exploited to provide a unification and generalisation of the modelling and design stages of the total structural design process. The relevance of, and motivation for using systems concepts in these contexts are examined along with the historical setting of the present approach. The emphasis at all times is on the development of a rational and systematic approach.

Using the conceptual foundation of a formal control systems theory and identifying the principal entities of state and control, a broad class of structural systems is reduced to canonical models and the components of a design problem are delineated. With the assumption of determinism, three distributed parameter system extensions of the maximum principle of Pontryagin are derived for three distinct system model types using a variational approach after Rozonoer, the technique of dynamic programming after Bellman and a classical calculus of variations approach. Singular formulations of design problems are identified for perhaps the first time. With the assumption of stochasticism, methodical arguments and Markovian properties are invoked to derive a set of conditions that an optimal design should satisfy. The results are extended to include reliability.

The treatment is confined to systems described by (in general) nonlinear vector partial differential equations in the four dimensional space-time domain. The concepts and results developed are clarified with illustrations taken from the first order elastic flexural (and shear) theories of beams, plates and shells.

ACKNOWLEDGEMENTS

The foundation concepts of Clyde (1970a, 1970b) and later Pister (1972) were used for the initial framework on which to base the present study. The author was indeed fortunate to correspond with Dr Douglas Clyde, Professor of Civil Engineering, Queen's University, Ontario, over the period of candidacy. Much of the research reported here is inseparable from the ideas of Clyde as expressed to the author over this period. Dr David Elms and Dr Peter Moss, respectively Reader and Senior Lecturer in Civil Engineering, The University of Canterbury, provided the invaluable local encouragement and technical advice.

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NOTATION

Symbols:

$\cup$	union
$\cap$	intersection
$\subset$	is a subset of
$\supset$	contains
$\in$	is an element or member of, belongs to
$\triangleq$	equals by definition, denotes
$=$	equals, is equivalent to
$<(>)$	less (greater) than
$\leq(\geq)$	less (greater) than or equal to
$/$	a symbol cancelled thus, denotes the negation of the symbol meaning
$\simeq$	equals approximately
$\forall$	for all
$\times$	cartesian product
$[a,b]$	closed interval $a \leq y \leq b$
(a,b)	open interval $a < y < b$
$(a,b]$	semiopen interval $a < y \leq b$
	rounded bracket implies strict inequality
$\delta(\ )$	Dirac delta
Π	product
$\sum$	sum
ext	extremum
max	maximum
min	minimum
$\Rightarrow$	implies
$\Leftrightarrow$	implies and is implied by
$\leftarrow \rightarrow$	to
$\lvert\ \rvert$	absolute value
$\oint_c$	integral over a closed path c, arc length σ, outward normal n
$\lvert J(\ )\rvert$	Jacobian determinant
∇	gradient vector operator $(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3})$

Braces denote a set or a family: for $e \in \Xi$

$\{e\}$ is the set whose generic element is e

$\{e;A(e)\}$ is the set with element e having property A.

A sequence $e_1, e_2, \ldots, e_m$ is denoted $\{e_i\}$ if the range of indices is clear.

$p(\cdot)$	probability density function
$F(\cdot)$	probability distribution function
$P\{\cdot\}$	probability of an event
$M\{\cdot\}$	expectation operation
$D\{\cdot\}$	variance operation
$P\{A\|B\}$	conditional probability; conditioned on the occurrence of the event B
E^m	m dimensional Euclidean space (Euclidean m-space) of m tuples with the norm of an element $e = \{\xi_1, \xi_2, \ldots, \xi_m\}$ defined as $\|e\| = (\sum_{i=1}^{m} \|\xi_i\|^2)^{\frac{1}{2}}$

$\partial_{\underline{\ell}} e$ with $\underline{\ell} = (\ell_1, \ell_2, \ldots)$, relates to a collection of partial derivatives of e, each with respect to $y_1, y_2, \ldots$ to the respective orders $\ell_1, \ell_2, \ldots$; the overall order of each derivative is $L = \ell_1 + \ell_2 + \ldots$, and $\ell_1, \ell_2, \ldots$ assume the range $0 \leq \ell_i \leq L$, $i = 1, 2, \ldots$ depending on the particular derivative. e.g. with only two parameters y_1 and y_2 and with $L = 3 = \ell_1 + \ell_2$ $(0 \leq \ell_1, \ell_2 \leq 3)$ this gives a collection of four derivatives

$$\frac{\partial^3 e}{\partial y_1^3}, \quad \frac{\partial^3 e}{\partial y_1^2 \partial y_2}, \quad \frac{\partial^3 e}{\partial y_1 \partial y_2^2}, \quad \frac{\partial^3 e}{\partial y_2^3}$$

The notation $\partial_{\underline{\ell}} e$ implies that one or more of these derivatives exists.

Conventions:

(i) All vectors are column vectors. Vectors are lower case underscored letters or derivatives with one element underscored. Components of vectors are subscripted.

(ii) Matrices are upper case underscored letters or derivatives with both their elements underscored. Components of matrices are doubly subscripted. (a_{ij}, i'th row, j'th column)

(iii) Differentiation, integration or other mathematical manipulations, when applied to vectors or matrices, are applied to all elements of those vectors or matrices respectively.

(iv) Scalar quantities are noted in the text and are not embellished except in certain cases where subscripts or superscripts are employed.

(v) Arguments of scalar-valued or vector-valued functions or functionals are placed in parenthesis or omitted where confusion is not possible. Equivalent notation is used to denote the value of a function at a given value of the independent variable(s).

(vi) Indexing variables are h, i, j, k, ℓ, α and β and take whole number values 1, 2, 3, ... Subscripts and superscripts may sometimes be used as mnemonics, but this form should be clear from the context. Superscripts L, R denote left and right interval limits; superscripts a, b denote boundary curves.

(vii) A variable with a superposed caret ^ is the optimal form of that variable in the sense of the context. Transpose is denoted by a superscript T. Inverse (reciprocal) is denoted by a superscript -1.

(viii) Probability: The conventional subscripts on the density and distribution functions indicating the random variable (vector) are omitted where no confusion is likely to occur. Only the realization will be given (in parenthesis following either the density or distribution function) to indicate the function. No distinction is made between a random variable and its realization. Context will be sufficient to distinguish the two. Finally, in the absence of limits, the integrations required in evaluating the expected values of variables will be taken to be performed over the whole real line E^1 (or E^n for an n-dimensional random vector).

Principal Characters:

c	probability density of the control; surface family parameter
D	spatial domain, boundary ∂D
E	Weierstrass function
f	function, state equation
F	augmented integrand §J; function §H, App.
g	function, end criterion
G	function, domain criterion
h	function, response equation, constraint inequation
H	Hamiltonian
I	augmented criterion
J	Rozonoer's functional
k	discretization index

L	Lagrangian
m	constraint counter; number of response coordinates
n	number of state coordinates; normal to curve, surface
N	total discrete intervals
p	end state counter; 'parameter' variable
q	end state counter
Q	optimality criterion
r	number of control coordinates; structure 'resistance'
s	dependent variable §F; surface areal coordinate
S	set of end states; limiting surface ∂S; surface
t	temporal coordinate
T	time domain
u	control variable
U	admissible region of controls
v	general dependent variable, App.
V	admissible region in state-control product space
x	state variable
X	admissible region of states
y	spatial coordinate; general independent variable
Y	time-position space; domain with boundary ∂Y
z	response variable
Δ	discrete interval
Λ	Lagrange multiplier §F
ζ	Lagrange multiplier
η	remainder term §F; local probability of failure; tangent plane vector §H; Lagrange multiplier §J
θ	angular (polar) coordinate
λ	adjoint variable; Lagrange multiplier §J
μ	reliability; Lagrange multiplier §H; sometimes control variable
ξ	sometimes state variable
ρ	cumulative probability of failure; radial (polar) coordinate
σ	switching function; arc length
Σ	piecewise continuous curve (closed), arc length σ, normal n
ϕ	function, side condition; angular (polar) coordinate
Φ	family of surfaces, parameter c
ψ	adjoint variable
ω	probability parameter
Ω	sample space

PART O

PRELIMINARY

§A INTRODUCTION

A.1 OVERVIEW

A.1.1 An outline of the approach. Despite advances in the development of sophisticated mathematical tools, confidence in the treatment of complicated structural systems, and the development of a certain order in the theory of structures, (whether in analytic or synthetic modes), there still exists very untidy thinking and lack of general understanding in both the modelling and design processes (Clyde 1970a, Pister 1972). Clarification of these processes is conceivable with the adoption of a format and thinking akin to systems theory (Klir 1969, Fel'dbaum 1965) where the aim is to provide a common basis and unified conceptual framework for studying system behaviour through generalisations and an ordering of knowledge. (The notion 'system' is used in the sense of sets of interacting elements or a transformation of 'input' data into 'outputs'.)

Modelling: The possibility of the formulation of mathematical models for structures which are meaningful and at the same time of wide applicability is offered through the medium of systems theory; fundamental to a systems theory is the establishment of suitable behaviour models expressing the interaction or interdependence of a system's components in a rational manner. Such a definite approach for structures has previously been obscured by their essentially associative (non-flow) nature (Gosling 1962). The bulk of systems concepts have been developed for sequential (flow) systems and hence the relevance (apart from weak analogies) to structural systems is not immediately apparent. Both flow and non-flow types, however, can be shown to share a common systems basis.

Design: As the modelling process in associated engineering branches has been restructured by systems approaches, so too has the design process been restructured by forcing an awareness on the designer to reassess his assumptions, goals and decisions (Hall 1962, Hare 1967). The approaches have enabled an objective approach to the formulation and the efficient solution of design problems. In these associated engineering branches where the service performance of the system controls the design process, the systems approach has proven to be powerful. It remains to show its validity in the field of structures, where a feedback learning cycle emanating from constructed designs is unfortunately absent, or at most piecemeal, and the service conditions are uncertain.

As suitable models are central to the conceptual foundations of systems theory, so suitable mathematical methods and thinking are central to the quantitative treatment of systems. In this respect, the philosophy of optimal control systems theory (Fel'dbaum 1965, Pallu de la Barriere 1967, Tsypkin 1971) will be useful for the structural design problem, as it exploits the composition of the system design problem.

An hierarchical multilevel system representation for structures:
The concept of a system provides a representation of behaviour through an assemblage of interacting subsystems. In this sense, a structure may be regarded as an hierarchical multilevel system (Mesarovic et al 1970, Mesarovic 1971), with the subsystems corresponding to the structure, member, element and material levels (figure A.1.1). The equivalent model description (input-output rule) of subsystem properties is the constitutive relationship. When combined on the next higher level with subsystem interaction (namely compatibility and equilibrium relationships), the three sets of relationships are then sufficient to define the behaviour on this next higher level (figure A.1.2). That is on any given level, the behaviour is studied in terms of that level's constitutive relationship while the manner in which subsystems on that level interact to form a higher level system is studied on the higher level. The foundation works in this field are Clyde (1970a, b) which also contain motivational material for the systems view.

With reference to figure A.1.1, subsystem boxes indicate the constitutive relationship at the given level. Three notions, namely control, state and response, have been introduced here and require explanation. For the present purpose qualitative explanations will be sufficient; complete definitions will be given later. Subsystem controls relate to the properties concerning components of the subsystem and the distribution of these components over space and time. The state refers to system internal behaviour, and the response to external behaviour. A distinction between the 'inputs' is required; the given information on the state derives from an interaction with the environment (as is the response an interaction with the environment) whereas the control derives from the designer. For all systems problems,

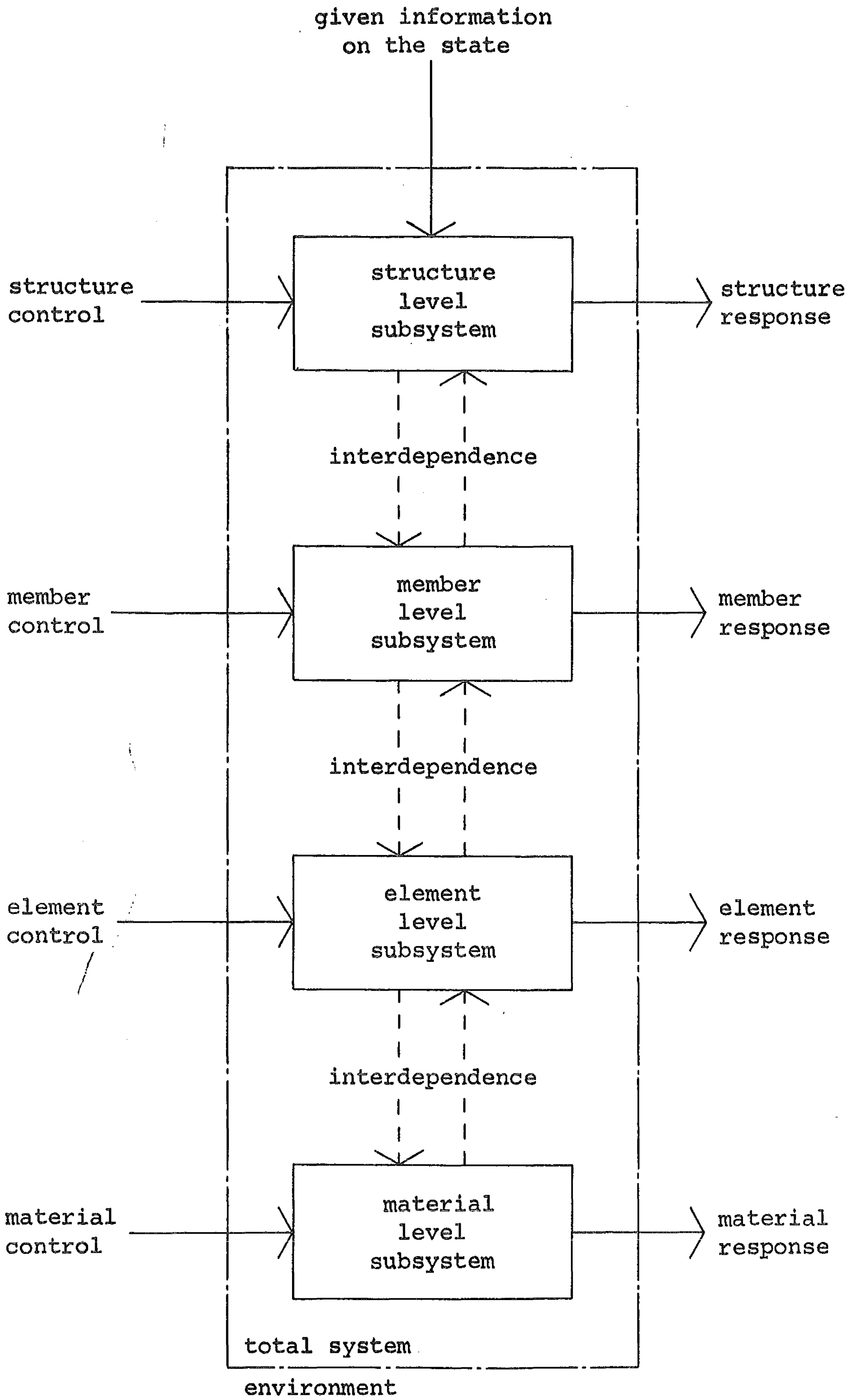

Figure A.1.1. An hierarchical multilevel representation

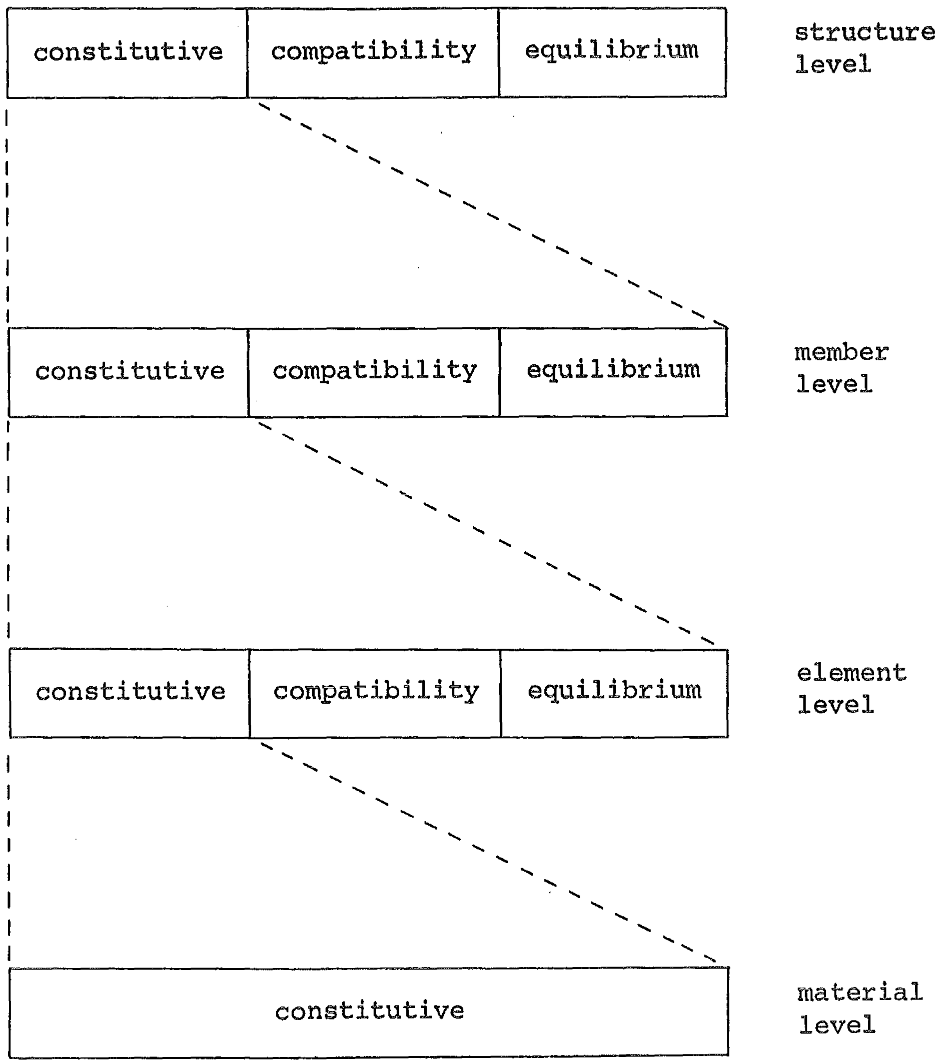

Figure A.1.2. Relationship between levels. A system on a given level is a subsystem on the next higher level.

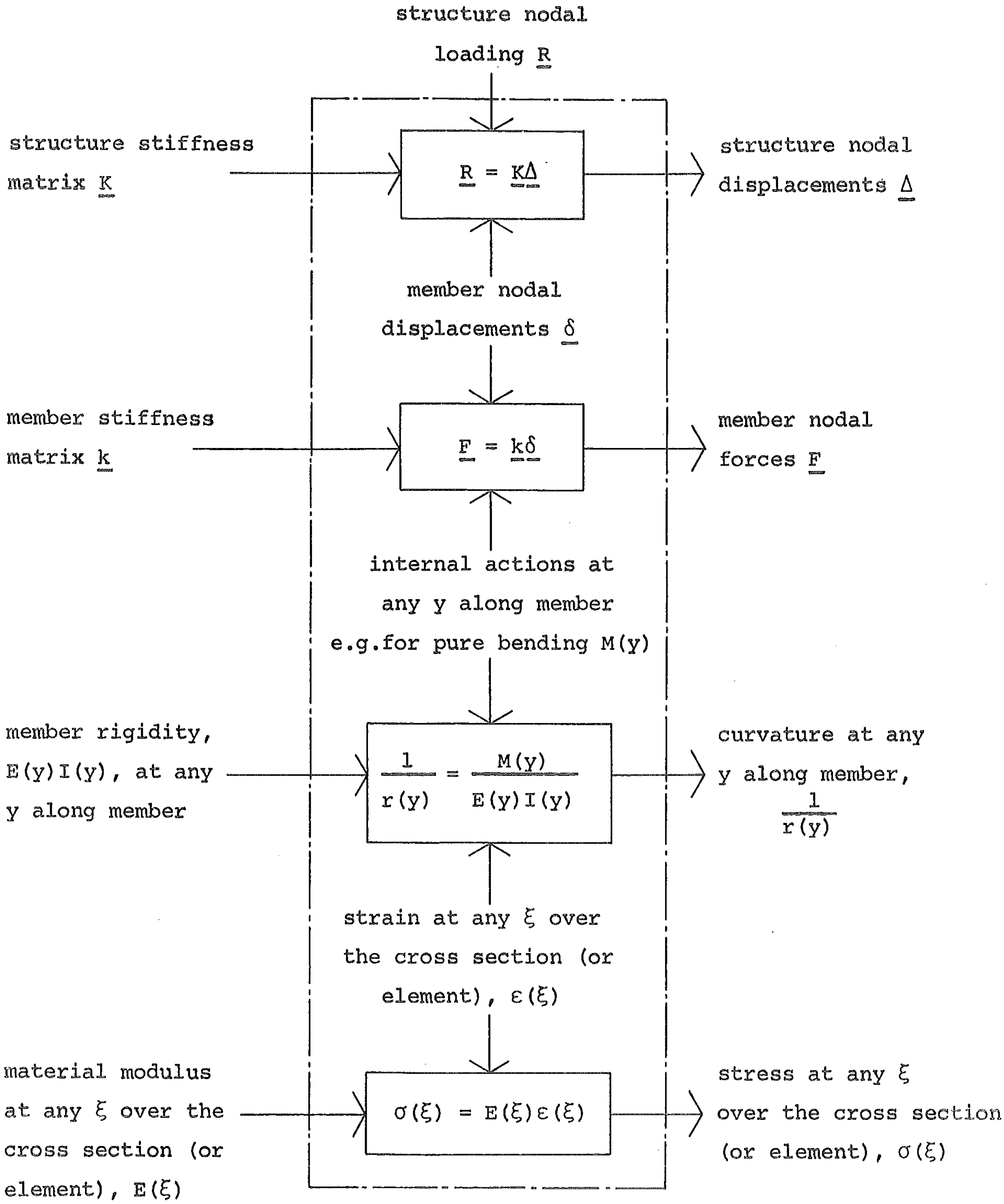

Figure A.1.3. Illustration of elastic modelling. Structure subjected to loading at nodal points.

the given state information is usually regarded as known whereas the control is free to be varied in the systems design problem case but given fixed for all other systems problems.

To fix the ideas, consider the illustration in figure A.1.3. This illustration is given in the sense of figure A.1.1 and follows the same illustration by Clyde (1970a) (given according to the sense of figure A.1.2). The illustration is for the case of structure nodal loading. Related schemes may be worked out according to the format of figure A.1.1 for other known information on the system state; for example, for the case of imposed structure nodal displacements, an 'inverted' scheme applies with the controls now becoming flexibilities in place of stiffnesses. Notice that there is an order of magnitude change of the information at each level.

The use of directed paths in figures A.1.1 and A.1.3 is purely schematic and implies dependence relationships and not flows of the entities. It facilitates the description of a system as sets of input-output pairs and is convenient when generalising systems concepts (which have primarily been developed for flow systems) to structures (non-flow systems). The use of the terms 'higher' and 'lower' when referring to levels is interpreted in the sense of the orientation of figures A.1.1, 2, 3.

It is seen that no levels are isolated; when considering any one level, the two adjacent levels must be taken into account. The interdependence between subsystems is indicated in figure A.1.1 by two-way arrows between boxes. The downward directed arrow represents information from higher levels that is needed to solve the lower level problem; upper levels define the bounds within which the lower levels function. The upward directed arrow shows that the construction (and behaviour) of the higher levels depends on the lower level construction (and behaviour). Control may be applied and exchanges with the environment may occur at all levels. Changes in controls on higher levels are manifested by parameter changes on lower levels. Understanding of the structural system functioning improves on ascending the hierarchy, while the detail unfolds on descending the hierarchy (figure A.1.2). Explanations of the total system behaviour are possible in terms of the lower levels and their interrelationships.

A single-level system study of modelling and design: Historically, systems theory has not been directed at multilevel systems in anything but a cursory fashion. As a result there is little understanding and framework on which to base an investigation of hierarchies, desirable as this obviously would be in the structures case where the system is fundamentally multilevel. Nevertheless by retaining a systems approach and working with a single-level system (figure A.1.4), some of the hierarchical nature of structures will unfold incidentally. At the same time, definite results may be obtained; results that would otherwise be unattainable at the present state of knowledge with a multilevel treatment. That is, emphasis in the present thesis is placed on the detail of a single-level system, the internal composition of this system and the relationship of this composition to the levels of a multilevel system. (The associated design problem considered is the single-level, single-goal (objective) problem, it being assumed that the design problem is formulated for a single objective. See §A.3.)

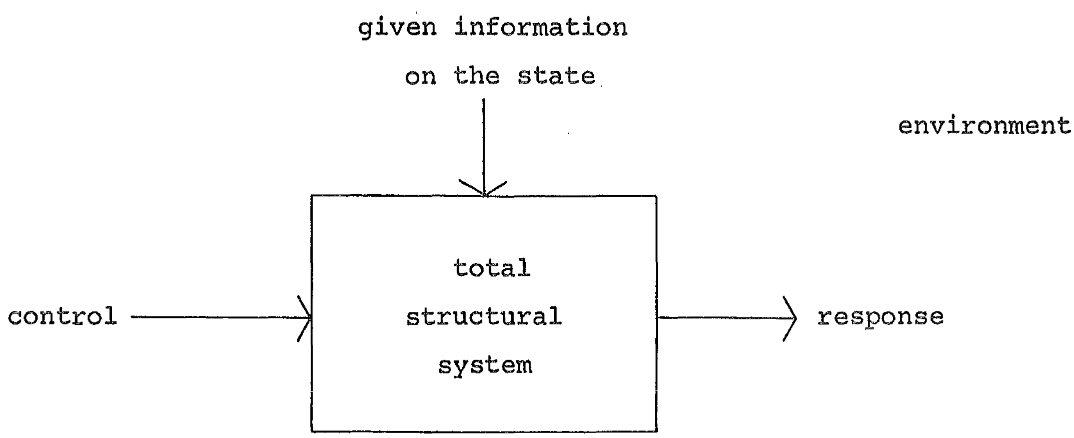

Figure A.1.4. Single-level structure representation.

The equations of conventional structural mechanics in their usual form, are regarded as unsuitable as a basis for a structures oriented systems theory and as a first step towards such a theory, a revised model is introduced. Three distinct entities-control, state and response - of the system may be identified. (The conceptual distinction has, in fact, existed all along, although it was frequently lost sight of in earlier work.) Conventional structural assumptions are not altered; the processes involved are one of reinterpreting the traits of the structure and one of reworking models. The choice of the revised model is founded on an examination of the principles of mechanics and an understanding of their inter-relationships. The model will permit ready statements of the fundamental systems problems of analysis (estimation), synthesis (design, optimization) and identification (investigation).

The final form of the present modelling work is a consequence of a desire to increase the understanding of the relationships of mechanics. Also ordered modelling in a suitable systems form is required before general systems design formulations can be achieved. Hopefully as a result of these two modelling aims, much of the untidyness that exists in current structural thinking (while acknowledging the advanced state of the theory of structures in both analytic and synthetic modes) may be put in order. Similar optimism has been expressed by Gregory (1963), Rozvany (1966), Brotchie (1967) among others.

As a direct consequence of the common modelling basis proposed for (a broad class of) structures, combined with a design philosophy identifiable to control systems thinking, rules may be derived for the single-level, single-goal design problem in terms of common characteristic properties. Such rules enable the collection of (a broad class of) structural design problems under a more generalized approach than conventional treatments in structural design. The difficulty in such general design endeavours is that the design problem formulation should be sufficiently broad so as to retain the necessary generality, yet narrow enough to permit effective solution. The form of the design results attained in this thesis is sufficiently general to be applicable to a wide spectrum of problems, yet capable of further generalisation.

<u>A.1.2 Thesis organisation.</u> (See §B.2 for a detailed outline of the thesis.) The presentation is logically divided into four main parts

following the introductory part (part 0; sections §A and §B). (A subordination consisting of 'part' (0, 1, 2, ...) → 'section' (§X) → 'article' (§X.x) → 'subarticle' (§X.x.x) has been adopted.)

The main objective of the remaining portion of this section and section §B is to introduce in a semiformal fashion, the necessary yet fundamental systems concepts, and to define the problems associated with the mathematical modelling of structures (§A.2) and structural design (§A.3). The relevance of, and motivation for using systems concepts in these contexts are examined (§A.2 and §A.3) along with the historical setting of the present approach (§B.1). Illustrations are used primarily as vehicles for conveying the essential ideas. They help to develop the arguments for using systems concepts in a structural sense and lead on to a generalisation and formalization of the results in part 1.

Part 1 lays the foundation for a systems theory interpretation of structures by assigning precise meanings to the fundamental systems concepts and discussing their properties in relation to conventional structural thinking. Both deterministic and stochastic formats are considered. The emphasis of the presentation leans in the direction of optimum design. The optimum problem is stated in the newly introduced systems terms and possible solution techniques are cited. (The solution of the optimum problem is taken up in detail in part 2 - deterministic design, and part 3 - stochastic design.) Illustrations are included to clarify the basic ideas.

Parts 2 and 3 develop the techniques and algorithms for dealing with the design problem within the conceptual framework offered by systems theory. The basic results are demonstrated in complete illustrations which also serve to highlight the basic modelling techniques of part 1.

Part 4 contains the appendices, references and closure with discussion on the approach advanced in parts 0, 1, 2 and 3 and on the direction in which future approaches may head.

The scopes of parts 1 to 4 are expanded in §B.2 following the introduction of the necessary terminology.

A.2 MODELLING USING THE FRAMEWORK OFFERED BY CONTROL SYSTEMS THEORY.

A.2.1 Basic notions. The conceptual framework offered by control systems theory as a rational basis for the modelling of structures is advanced. Essentially inductive arguments will be found most useful for this purpose. A particular example (the beam equation) is studied and certain physical and conceptual entities (control, state and response) are used to describe it. The utility of these entities as measures of generalisation and uniformity of structural description as well as their relationship to a design and understanding viewpoint is discussed. To extend the theory by general inference, all other related structural systems would be required to share these qualities. This they are shown to do readily, as the model is consistent with established structural mechanics assumptions. The utility of the theory in a structures context is thus shown, and this leads on to a generalisation of the modelling results in part 1. It is emphasized that it is only the conceptual framework which is borrowed from control theory as, in general, structural problems are far more complex than the form of application treated by the parent theory.

By way of introduction, several essential concepts of control systems theory may be identified:

Control variables, contain the information relating to the physical properties (flexural stiffness, axial stiffness, ...) of the various components of the system. They exert the control on the behaviour of the system and may be freely chosen (manipulated) by the designer. In this sense they are 'input' into the model by the designer.

State variables, contain all the information regarding the internal behaviour or state (deflections, induced moments, ...) of the system.

The system response indicates the level of outward behaviour (deflection, ...) of the system. Response variables may or may not be identified directly with state variables. Generally the response, being a relevant measure of the state, will comprise part of the state description.

It follows that the state relates the 'input' control to the 'output' response and determines the response uniquely for a given control. Both state and response are controlled variables. In this context, only controlled systems are implied; that is the system state (and response) may be manipulated by careful selection of the control. The term 'controlled' implies a causal relation between control and state (and response). A system as mentioned in the previous article may be conveniently thought of initially as an interaction of structural elements or input-output transformations. A more explicit meaning is not required at this stage. A collection of elements is insufficient to describe the system; interaction has also to be specified. The environment may be taken to include everything not defined as the system. The environment affects the system by changes and is affected by changes in the system. (This definition follows Hall 1962.) In conventional structural calculations part of the environment is usually replaced with, for example, load conditions, enforced structure displacements and others which are known for any given environment-system arrangement. Another form of system-environment interaction are the usual notions of boundary and terminal conditions.

Following from these definitions of system and environment, it is apparent that any given system may be reduced to subsystems. The hierarchical multilevel system of figure A.1.1 (or A.1.2) follows, with the subsystems delineated at each level of behaviour. The notions of state, response and control extend to all levels (Clyde 1970b) such that materials, dimensions, geometry, rigidities, ... may all be considered controls while stresses, strains, curvatures, ... may all be considered states at various levels of the total structure hierarchy. At the lower levels the response and state may obey a one to one transformation and the distinction in this case between response and state disappears superficially. However for consistency of terminology on all levels, the distinction will be maintained. The concept of state relates best to systems variable in space and/or time, and to which a control is introduced and a response (or output) calculated.

The fundamental systems problems of analysis (estimation), synthesis (design, optimization) and identification (investigation) may now be introduced in relation to this terminology. (See for example Lee 1964, Klir 1969.) Analysis procedures regard the control variables to be given, with the only true variables being the state variables. Synthetic type techniques attempt to assign the control variables so as to give a desired state (direct synthesis) or to optimize some design goal (optimal synthesis, optimal control, optimization). Both the analysis and synthesis problems require the a priori specification of the system model; however identification procedures determine the form of the system model for given input-output characteristics. They are commonly called 'black box' problems, implying a complete or partial lack of knowledge of the organisation of the model. The theory of control embraces these fundamental problems. The relevance to the theory of structures is apparent.

To demonstrate the notion of state, consider the Bernoulli-Euler beam constitutive relationship of the form

$$\text{(a.2.1)} \qquad \frac{d^2}{dy^2}\left[D(y)\ \frac{d^2 w(y)}{dy^2}\right] = q(y) \qquad\qquad y \in [y^L,\ y^R]$$

For a given control D (denoting flexural rigidity), and loading q, the solution of this equation (the response displacement w) is completely resolved at any position $y \in [y^L, y^R]$, when a combination of four - depending on the individual problem - of the total possible static (stress) and/or kinematic (displacement) boundary conditions are specified (shared) between y^L and y^R. It will be recalled that the kinematic boundary conditions relate to

(a) the deflection w, and

(b) the rate of deflection, $\frac{dw}{dy}$,

while the static boundary conditions relate to

(c) the internal moment (proportional to the second derivative of w), and

(d) the internal shearing force (proportional to the third derivative of w).

These four quantities will be referred to as the <u>state</u> of the beam at y^L and y^R. However it will be noted that the interval limits y^L and y^R are arbitrary, and in fact the four quantities may be specified at any position y for a solution to be gained of (a.2.1). Hence the state could be defined at each position y along the beam. In this sense the state (at any y) contains the minimal amount of information required to determine the state at some other position y', for any given control.

Provided the state is known at any position y, the response may be evaluated for a given control; the state relates the input control to the output response and uniquely determines the response for a given control. It is the necessary information required to determine the response completely. The boundary conditions (or 'end' conditions, implying a visualization of the interval $[y^L, y^R]$) are known values of state at certain positions along the beam (here the left and right ends) and are sufficient to define the values of state for all y. (For a more general structural theory than considered here, loading (surface traction or static boundary condition) too may be considered as an 'end' condition on the states which also become more general than considered here. Boundary conditions (static and kinematic) represent an interaction of the system and environment as mentioned earlier.)

The state may alternatively be thought of as representing the internal behaviour of the beam. As such it would then appear reasonable that to exercise control over the system, information about the state would be more useful than a single quantity representing the gross or outward system behaviour. (In this case the response is the deflection w.) Established structural practice eliminates all but one 'behaviour variable' (generally the response variable) and proceeds to solve for this (in the form of (a.2.1)); systems concepts define additional behaviour variables (for example quantities (a) → (d)) that incorporate this one variable (for example quantity (a)) and are collectively denoted as the state. The outward behaviour follows straightforwardly from a knowledge of the state and control.

The difference between working with the internal behaviour of the system (that is the state) and the outward behaviour (response) will be seen to be twofold. Firstly as discussed above, control may be applied more efficiently by working internally to the system and increasing the flexibility of the designer/system interaction. Secondly, it will be shown that the state concept appears more rational as it exploits the basic composition of the structural system equations (in this case equation (a.2.1)). The hierarchy proposition of Clyde (1970a) will then be seen in a still more favourable light. Structures may be adequately described by their outward behaviour characteristics, but for proper functional understanding a description in terms of lower levels is required.

A.2.2 The formulation of state models; the state space. For the beam example presented above, having decided on the choice of state variables parameterizing the input-output transformation (a.2.1), this same equation may be reworked slightly and interpreted in a standard form which will be adopted exclusively for the remainder of this thesis. The advantages of the standard form will be apparent and are emphasized following its introduction.

Corresponding to the state quart-tuple, new variables $\{x_i;\ i = 1, \ldots, 4\}$ are introduced

$$(a.2.2) \qquad x_1 \mathrel{\hat{=}} w \qquad x_2 \mathrel{\hat{=}} \frac{dw}{dy} \qquad x_3 \mathrel{\hat{=}} D\,\frac{d^2w}{dy^2} \qquad x_4 \mathrel{\hat{=}} \frac{d}{dy}\left(D\,\frac{d^2w}{dy^2}\right)$$

which define a four dimensional state space in which $x_1, \ldots, x_4$ are coordinates. Then by differentiation and with control $u \mathrel{\hat{=}} D$, equation (a.2.1) may be written as four simultaneous first order equations. Isolating each derivative $\frac{dx_i}{dy}$ on the left hand sides;

$$(a.2.3) \qquad \begin{aligned} \frac{dx_1}{dy} &= \frac{dw}{dy} = x_2 \\ \frac{dx_2}{dy} &= \frac{d^2w}{dy^2} = \frac{x_3}{u} \\ \frac{dx_3}{dy} &= \frac{d}{dy}\left(D\,\frac{d^2w}{dy^2}\right) = x_4 \\ \frac{dx_4}{dy} &= \frac{d^2}{dy^2}\left(D\,\frac{d^2w}{dy^2}\right) = q \end{aligned}$$

Equations (a.2.3) are equivalent to the original system equation (a.2.1). Equations (a.2.3) will be referred to as the state equations (or system model equations or system equations) and belong to a frequently used standard (canonical, normal) form.

(a.2.4) $$\frac{d\underline{x}(y)}{dy} = \underline{f}[\underline{x}(y), \underline{u}(y), y]$$

(See for example Fel'dbaum 1965, Athans 1966.) In (a.2.4) the state and control variables are clearly distinguished. (In general an n'th order differential equation will require n quantities to specify the state of the system model and n first-order equations will result.)

The response will also be taken to conform to a standard form (though now algebraic rather than differential) referred to as response (output) equations

(a.2.5) $$\underline{z}(y) = \underline{h}[\underline{x}(y), y]$$

(See for example Porter 1969, Lee 1964.) For the example above, the response w equals the state x_1 directly and is clearly a special case of (a.2.5).

The general form of (a.2.4) and (a.2.5) may be obtained from intuitive arguments on the roles played by $\underline{u}$, $\underline{x}$ and $\underline{z}$ in the description of any system. Equations (a.2.4) and (a.2.5) (together with listed end-state conditions) specify the model of the system (abbreviated to (state) model or system where no confusion may occur).

Certain characteristics of equations (a.2.4) and (a.2.5) will be apparent. The right hand sides only contain the state variables $x_i(y)$, controls $u_j(y)$ and constants of the system such as q(y). $\underline{f}$ and $\underline{h}$ are vector-valued functions of the state vector $\underline{x}(y)$, control vector $\underline{u}(y)$ and independent variable y. System constants (for example loading and frequently materials and certain geometry terms when not required as controls) are omitted from the standard form of (a.2.4) and (a.2.5). In general these constants will be given for any given situation. The state variable form corresponds with the so termed normal form of the theory of differential equations and the equations are said to be normalised (Pontryagin 1962).

The representation (a.2.4) is advantageous in that it allows a common description of all systems (of the class considered) while standard numerical solution techniques of differential equations, techniques which usually are only valid for first order equations, may be invoked.

Restricting the discussion to (a.2.4) for the moment (with the knowledge that the response $\underset{\sim}{z}$ follows straightforwardly from (a.2.5) for given state) certain points require amplification. Derivatives of the state (only) appear only on the left hand side. The state is assumed to be defined for a given control $\underset{\sim}{u}$ belonging to an admissible class of functions. (The matter of mathematically admissible functions is discussed in a later article, but for the present, it is noted that normal engineering structures satisfy the admissibility requirement on $\underset{\sim}{u}$.) The state at any position y of a system defined by the n differential equations (a.2.4) may be represented by a point in the n-dimensional state space. (The state space is the set of all $\underset{\sim}{x}(y)$). The locus of these points describes the state over the y interval.

This decomposition process of the constitutive relationship, equation (a.2.1), illustrates a general procedure to be followed in the remaining presentation; namely the introduction of simpler equations ((a.2.3)) which may be cast into a general system model class (equations (a.2.4)). Note that the choice of the detail in the simpler equations is not unique but in the present context the transformations will be introduced such that the resulting first order differential equations have a well-defined physical significance in terms of relationships of equilibrium, compatibility and constitution. This breakdown is applicable for static behaviour and dynamic behaviour where space derivatives occur on the left hand side; clearly for dynamic behaviour and with time derivatives chosen on the left hand side, the equations will assume a different interpretation (though still meaningful physically) but this will be discussed later in this article. (The development of a suitable mathematical model, being the first step in the analytic or synthetic treatment of structures is critical. Any choice of variables satisfying the mathematics of (a.2.4) would have been acceptable. The lack of physical significance of the variables and equations involved in this case however holds little intuitive appeal although computationally it may be advantageous. The reader is referred to section §C for amplification of these comments.)

The three basic relationships (equilibrium, compatibility and constitution) in the above beam example are readily identified: (i) equations $(a.2.3)^3$ and $(a.2.3)^4$ comprise the equilibrium relationships while (ii) equations $(a.2.3)^1$ and $(a.2.3)^2$ comprise compatibility and constitution. (It will be appreciated that the states x_1 and x_2 are the conventional generalised displacements while the states x_3 and x_4 are generalised forces.) For statically determinate structures (i) and (ii) are uncoupled as anticipated and may be solved independently. Conversely (i) and (ii) are coupled in statically indeterminate structures. In general, however, no distinction is necessary between statically determinate and statically indeterminate structures, the latter being the general case. For dynamic problems (and space derivatives chosen on the left hand side) the equivalent sets of data are (i) equations of dynamical equilibrium and (ii) equations of kinematics and constitution.

Notice that for systems interpreted in the form of equations (a.2.4) and (a.2.5), the underlying mathematical structure is the theory of vector spaces and the space (and time) domain treatment of vector differential (and difference) equations. The illustration covered systems described by ordinary differential equations (lumped parameter systems). The concepts extend readily to systems governed by partial differential equations (distributed parameter systems). By defining suitable function spaces, the formalism of equation (a.2.4) can be shown to be applicable to these types of systems as well (Katz 1964, Yu. V. Egorov 1963, 1966, Falb 1964, Balakrishnan 1963, 1965, for example). The spaces in this case are not finite dimensional. However in the present work, the initial axiomatic-type presentation of modelling (part 1) will retain the partial differential form. Equation (a.2.5), being algebraic, extends directly to the distributed parameter case.

A.2.3 Comment. The single-level model permits a ready application of state concepts because of the known internal make-up in terms of equilibrium, compatibility and constitutive relationships (the last defining the subsystem at the next lower level of an equivalent multilevel model).

The introduction of the concept of state, in conjunction with established ideas of structural mechanics, offers a framework on which to base a much deeper study and understanding of structural behaviour and the foundation of a useful structures systems theory. It will be shown that many problems, previously considered complex and intractable, may now be given a solution. Notice that the established ideas of structural mechanics have not been cast aside but have been integrated into the concept. Conventional structural assumptions are not altered. However the structure components are reinterpreted and the equations reworked into a form which is now suitable as a basis and a unified conceptual framework on which to develop a structures systems theory.

The concept of state for the dynamic case (time derivatives chosen on the left hand side) may be regarded as an axiomatization of Newton's laws of mechanics (Kalman 1962, 1963a); the inference of defining momentum as the product of mass and velocity such that mass times acceleration is then the rate of change of momentum, has an analogous control theory interpretation. For example, a vibrating lumped mass;

$$\frac{d}{dt}\left(m\,\frac{dw(t)}{dt}\right) = F(t) \qquad\qquad t \in [t^L, \infty)$$

The motion is uniquely determined for all t, if, given the control m(mass) and forcing function F(t), the entities w and $m\,\frac{dw}{dt}$ are specified at any time t. The value of t at which these two entities are specified is immaterial. The state is thus the position - momentum pair $(w,\ m\,\frac{dw}{dt})$. This may be taken as the basic definition of the system. Thus, for deterministic systems, the state at any time t is the minimal amount of information needed to completely determine the behaviour (state) of the system for all other times for any given control.

To extend the concept of state to stochastic systems, the state at any time t is regarded as the information that uniquely determines the probability distributions of behaviour (state) at all other times. By definition this describes a Markov process (Bharucha-Reid 1960, Wong 1971, Prabhu 1965, for example). This definition for state in stochastic systems is a basic assumption and implies a form of dependence between adjacent states (but not total dependence between all states). In general the system will not have states with these properties, but the assumption allows analogous treatments between the deterministic and stochastic cases. (Markov processes are the stochastic equivalent of differential equations.) The assumption enables solutions to be found that would otherwise not be possible if full stochastic dependence of states was employed. For an introduction to state space concepts in stochastic systems see Fuller (1960b).

For the dynamic case of classical mechanics, (for example Synge 1960, Lanczos 1949), for a system of particles or rigid bodies with k degrees of freedom, the state space (usually known as phase space) is 2k-dimensional with coordinates $q_1, \ldots, q_k$ and $p_1, \ldots, p_k$ where q_i and p_i are generalised coordinates and momenta respectively. The equations of motion (using a variational approach rather than the equivalent Newtonian view) can be described by a set of 2k first order differential equations (Hamilton's canonical equations - they are canonical adjoints):

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

(a.2.6)

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad i = 1, \ldots, k$$

where H is the Hamiltonian. The motion is then the path of a point in phase space. (Occasionally time may be included in the definition of phase space, in which case the phase space is the cartesian product of the above phase space in E^{2k} and the time space ($= E^1$).) In essence the approach is to associate with a system a set of coordinates which define a space, of dimension equal to the order of the system. The behaviour of the system is then represented by a trajectory in this space.

Control theory uses a simple generalisation of this. In transferring to the analogous state space in control theory it is noted that the concept of degrees of freedom does not transfer directly yet the concept of phase space does. It is the concept of phase space which has been generalised to the definitions of state space used in control theory. See Fuller (1960a) for an excellent historical survey of state space concepts, and Zadeh and Desoer (1963) and Tou (1964) on the concept of space. Additional to the idea of state, control theory separates out the control from the other system entities and often interprets this as an input which is varied to give the system a desired response.

Inductive reasoning extends this dynamic basis for the idea of state to the case with a spatial coordinate as the independent variable. Whereas in the dynamic case only positive time has meaning, by suitable definition of axes, both positive and negative spatial coordinate values are possible.

Concepts of state are also known in other branches of applied mathematics and theoretical engineering. They are, for example, an integral part of the field theories of continuum mechanics (for example Truesdell and Toupin 1960), the mechanical behaviour of materials (for example Freudenthal 1950), and the transfer matrix theory of matrix structural analysis (for example Pestel and Leckie 1963, Livesley 1964). Their use in control theory however appears to be only a relatively recent innovation arising from generalised circuits theory and the above dynamic basis. (Terminology, concepts and symbolism have also been generalised from these two roots wherever needed.) Their use in the present work generalises completely the use of state in a structural mechanics context by adopting the framework of control theory.

The illustrations and the development of the theory in this thesis are offered on a theory of structures level corresponding for example to that involved in reducing the complete continuum mechanics equations to the beam equation. For the full implications of the proposed control systems theory treatment, a general continuum mechanics approach of the form sought by Pister (1972) (or a slightly more specific four-dimensional theory of elasticity (for example Sokolnikoff 1956) approach) would be desirable. The present approach based on a simplified theory of structural behaviour, while losing generality and producing slight inconsistencies (for example surface tractions are state boundary conditions but the approximate form of the beam equation (a.2.1) has already eliminated the corresponding state variable and the load there is treated as a constant) does produce tangible results, results which it is felt would be unattainable using a more general theory. Fung (1969), among others, shows the connection between a general and simplified structural theories.

A.3 DESIGN IN A SYSTEMS SENSE.

A.3.1 General. An examination of the total structural design process may be profitably undertaken using the approach of 'systems engineering' (in the sense of Hall 1962, as distinct from a systems theory or a control systems theory) with its methodology based essentially on generalisations of real case histories. Established structural design procedures are seen to be iterative in nature. The iterations arise from the analysis-based mode of attack on the design problem and are not inherent in design. By suitably defining the design problem, it is shown that much of the iterative process of established structural design procedures may be eliminated if emphasis is placed on a synthetic approach. Generally a structure will be synthesized in an optimal sense, with the optimization being performed in terms of a criterion derived from imposed (often subjective) value statements. Using the modelling procedures of the previous article, it is shown that the optimum design problem is now within the realm of the well delineated body of theory and techniques of optimal control systems. In this sense the design problem is a single-level, single-goal problem.

A.3.2 The design process. The logic of the evolution of a design may be conveniently interpreted (Hall 1962) in the six stages of problem definition, value system definition, system generation, system evaluation, selection and action. The order of attack is as critical as the development of a rational system model and strongly influences the final design. By providing such a construction from which to work, each stage may be given a correct perspective (Khachaturian 1968, Clyde 1970a). Feedback may occur within stages in an effort to refine the problem at any of the six stages, while a certain merging or overlapping may be noticeable between successive stages. A systematic approach to the hierarchy of stages in the design process will generate clear thinking at each stage and lend objectivity to a procedure which would otherwise be considered qualitative or intuitive.

Expressed in systems terms, the conventional notion of 'structural design', being a phase of the total design process, may be viewed as a closed loop operation of iterative modification and feedback to the analysis stage (designated by the full lines in figure A.3.1). The terminal points of the loop cycle are based, respectively;

(a) initial, upon a postulated system extrapolated from experience or based on imagination (third stage of the total design process), and
(b) final, upon satisfaction of a prearranged performance specification (fifth stage of the total design process).
In short, conventional design is a process of trial and error optimization. Very lucid discussions on the philosophy of design may be found in Gregory (1963), Pister (1972), and Porter (1969).

For many structures the stage of system generation is routinely obvious with the subsystem interrelations predefined by mechanics. Systems are commonly divided into lower level subsystems (in analogy with 'subsystem delineation' of Hall 1962) to produce a tractable model and a tractable design subproblem, although certain inconsistencies in the modelling procedure will be noticeable (Clyde 1970a). In particular there exists an interdependence of each level, requiring knowledge at a higher level when designing a lower level subsystem. No isolated systems exist. The introduction of a series of design subproblems creates further iteration in the design process (that is further to that produced by an analysis-based approach). This iteration again is not inherent in design.

A.3.3 Synthetic transformation within the design process. The iterative nature may be partly removed from the design process, if for specified requirements of a design, the system is synthesized directly to meet the specification, the operative word being 'directly'. The essential difference between the analysis-based and synthesis-based procedures is at the level of abstraction adopted in the computations. (The terminology 'level of abstraction' is used in the sense relating to the quantity of a priori data assumed.) Analysis-based techniques impose a total system configuration ab initio, while the system emerges from any given level of abstraction as a natural consequence of the direct synthetic treatment. Presumably the extreme generality that may be attained in the direct case would involve little or no a priori knowledge of the emerging structure - refer level of abstraction (A) in figure A.3.1. However for a solution of practical significance, certain leading properties of the system configuration are best assumed - the corresponding level of abstraction is intermediate between levels (A) and (B) in figure A.3.1. The choice of abstraction level on which the designer chooses to work

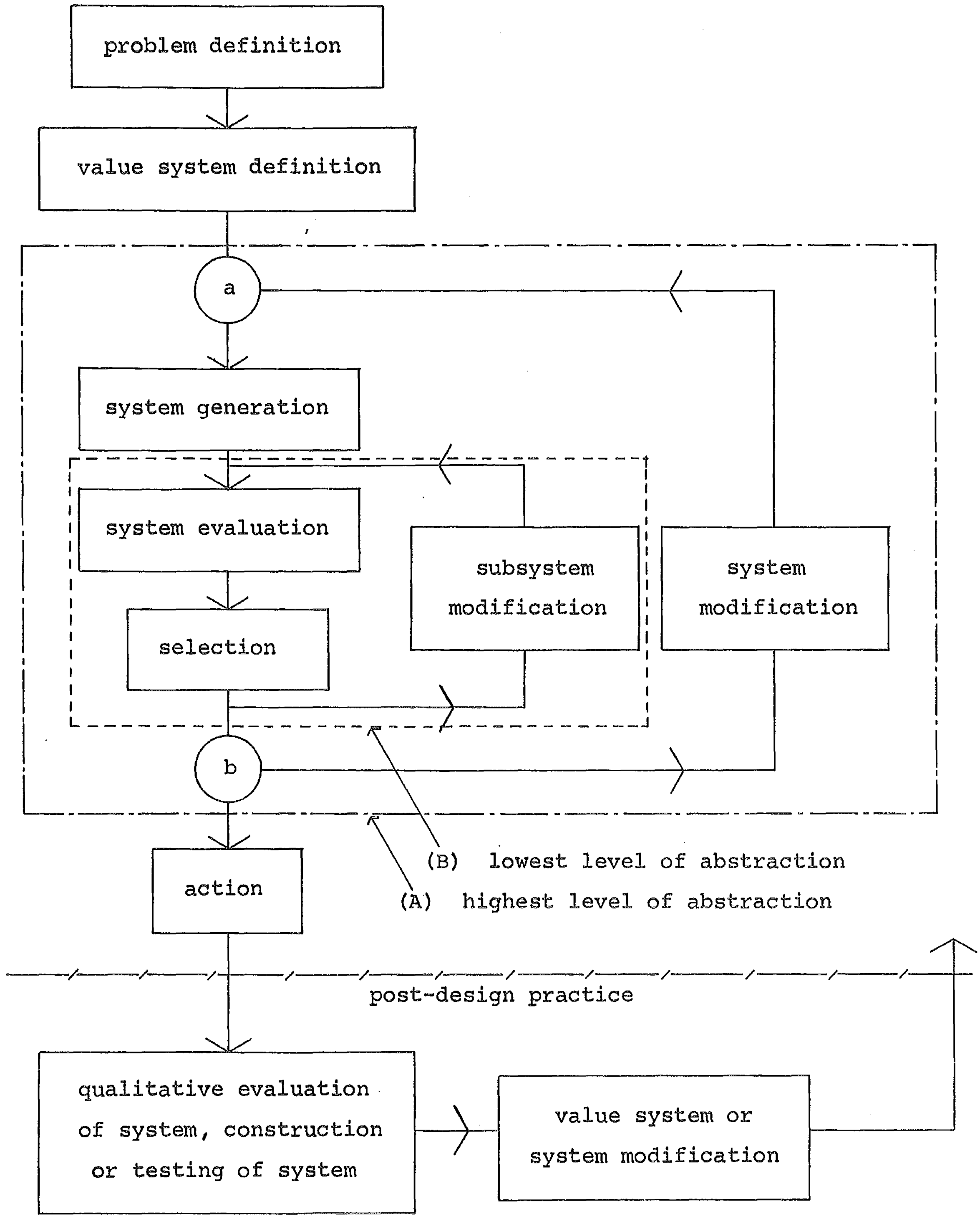

Figure A.3.1. Structural Design Process

would be a balance between his engineering judgements and desired computation load. A synthesis-type treatment can only proceed where certain of the system properties remain free and adjustable. It is apparent that a synthesis-type format to design is the fundamental and at the same time more rational approach. Brotchie's (1967) systematic interpretation of the design problem exemplifies these comments.

A.3.4 The theory of optimal control systems. Generally one desires to synthesise a system which is optimal in a certain sense; a design criterion or index, resulting from an imposed value statement, is implied. Optimum system design is of central concern in optimum control theory which exploits the synthetic nature of the problem. It is the philosophy of this theory that will be found most useful in the structural design problem. The philosophy rests on very broad grounds, typical of techniques in systems theory, only conversing in the entities state and control, which take on very definite meanings in the design problem.

In simple terms, synthesis is thus equivalent to choosing the controls throughout the structure; optimal synthesis or optimal control selects the controls so as to extremise some design criterion. In addition supplementary design constraints are also usually present. It is the theory of control that is concerned with the mathematical formulation of laws for the control of systems. These comments are amplified in part 1 following the introduction of the necessary terminology.

Optimal control theory in other engineering branches has elevated the 'art' of design to a status approaching a systematic and exact 'science'. This has occurred despite the everpresent yet necessary 'engineering judgements', which recognise the existence in all designs of certain intangible quantities that defy precise mathematical statements. This theory offers the same advantages to structural design.

§B BACKGROUND AND SCOPE

B.1 BACKGROUND.

B.1.1 The design problem. The fervour of research in the field of system optimization in recent years has yielded a theory with a certain amount of maturity. Certainly realistic problems are now tractable using the concepts and techniques offered by 'modern' control theory.

Historically, the aerospace, electrical and chemical engineering disciplines have promoted the development of this more substantial theory in an attempt to satisfy more stringent design specifications and operating conditions. The merging of ideas in these disciplines under a general control systems theory has stimulated the recent research impetus through drawing upon the results and experience in these separate branches of engineering. Modern control theory, in these disciplines, has supplanted design tools that were largely graphical or qualitative in character and a design process of a trial and error nature. It is now firmly entrenched in these disciplines as a result of its proven utility in design. The introduction of the digital computer may be partly credited with this change in design philosophy.

The related (in fact the present work shows that it is subsumed) field of structural optimization, however, has tended to proceed along established structural design avenues, with the result that progress has been slow and the level of realism of the problems tractable has been all too low. Nevertheless, the field has received heightened interest in recent professional journals. A review of the field may mention the surveys of Wasiutynski and Brandt (1963), Barnett (1966), Gerard (1966), Kowalik (1966), Rozvany (1966), Schmit (1969a, 1969b), Sheu and Prager (1968) and Pierson (1972) among others; the monographs of Cox (1965) and Owen (1965); the texts of Fox (1971) and Spunt (1971); and a recent AIAA film (1972) on the subject. Certainly publications in the field have been very profuse, and while the 'state of the art' reviews above concentrate much of the essential literature, they are far from exhaustive.

From these works in structural optimization, two trends may be recognised;
(a) the development of a formal optimization theory where general results are established, and

(b) the direct recourse to numerical methods for individual problems. Entries in the latter category are by far the most abundant. The treatments are essentially exercises in mathematical programming and related extra-discipline optimization techniques.

The intention here is not to review the total structural optimization scene per se, a formidable task in itself, but rather the development of optimization practices specifically based on the methods of control systems theory.

Analogies between structural optimization problems and problems in optimal control (and also multistage decision theory) have recently been employed to give an extended range of optimization solution techniques. However there appears to be no realization of the significance of their analogies (in terms of modelling of their structure) apart from a direct application of the mathematical techniques to the optimum problem:

(i) Comparable solutions to the classic structures of Michell theory were derived by Goff (1966) by reordering the problem in a decision theory format. The dynamic programming technique was used in the solution. Related work on skeletal structures may be found in Palmer and Sheppard (1970) and Sheppard and Plamer (1972). Distefano has applied the same technique to rotating disks (1972).

(ii) The equivalence of the mathematical structures of the problems in optimal control and structural optimization has been applied effectively by Dixon (1967; see also 1968, 1972; see also Boykin and Sierakowski 1972 for discussion), Ashley and his coworkers at Stanford (Ashley and McIntosh 1969; McIntosh, Weisshaar and Ashley 1969; Armand and Vitte 1970; Weisshaar 1970; Armand 1971, see also 1972), Citron (1969), de Silva (1972) and Bellamy and West (1969). The last reference uses analogue simulation to solve the optimization equations resulting from the use of the maximum principle, while the remainder adopt numerical techniques adapted to digital computation whenever closed form solutions are unattainable. Related interesting work may be found in Haug and Kirmser (1967) from which have come Haug (1969), MacCart, Haug and Streeter (1970), and Haug, Pan and Streeter (1972). Various optimal control algorithms are employed.

Of the above several contributions stand out. Armand's work (1971) would appear the most enterprising contribution, particularly in terms of treatment of complex systems. In fact his work is the only one to date dealing with distributed parameter problems. All other references treat the far simpler lumped parameter problem; their level of presentation is similarly restricted. Armand's formulations can be shown to be singular (see section §L) and hence his solutions are not necessarily minimizing in the sense he claims. (His solutions are nevertheless optimal despite not recognising the singularity.) Similar comments apply to de Silva (1972). Both Citron (1969) and Haug and Kirmser (1967) treat inequality constraints on the state in their designs. In general, state constraints complicate the computations inordinately and are usually avoided by most authors. In much of the work from Stanford on free vibration problems, equality constraints are handled by direct substitution of the appropriate constraint in the system equations. The systems are then without loading and unconstrained. In such circumstances the state and auxiliary solution (adjoint) variables can be shown to exhibit a simplifying property in that they are constant multiples of each other (see Sections §G, §I, §K, §L) and the solution computations, though still nontrivial, are relatively uncomplicated. However for most design problems, this simplifying property is not apparent from inspection of the equations.

So far mention has only been made of structures behaving in a linear elastic fashion. Here the equation describing the structure (the system equation) is the constitutive relationship at the appropriate level. However in the plastic regime of behaviour, and particularly when a lower bound formulation is used in design, the system equation is the equilibrium equation. (It is noted that the equilibrium equation is a relationship between state variables and hence contains no variables that are control variables in the conventional design sense. In this form the system is theoretically uncontrollable. To counter this, the dependent variables in the equilibrium equation may be arbitrarily classified as control and state variables and the solution follows conventional state-control treatments. The mathematics can still handle the problem; it is only the conceptual basis which is lacking.) Both Price (1971) and Lepik (1973) discuss plastic design in a state-control format. Price uses linear programming and Lepik, the maximum principle. Palmer (1968) employs dynamic programming for the design of elastic, perfectly plastic continuous beams.

The notions of state and control are applicable to structures behaving in accordance with any other form of constitutive relationship although no work has been reported in the literature. Obviously the optimization problem for the linear elastic case is the most mathematically tractable and hence has been the initial emphasis.

(iii) Passing to the conceptually more difficult stochastic case, the ideas of dynamic programming were first suggested by Kalaba (1962) as a means of designing determinate structures for a given reliability. For trusses, although in no way serial or sequential in nature, the member geometries are chosen in a pseudo sequential manner using Bellman's principle of optimality. Khachaturian and Haider (1966) and Khachaturian (1968) have followed up this initial work by applying it to a determinate truss. No other applications of analogous control techniques to probabilistic design appear to have been published. This may be attributable to an only partly common mathematical structure between structural design problems and optimal control problems in the stochastic case. The mathematical structure in the deterministic case by comparison is fully common.

To summarize; the earliest established analogies appear as essential steps in the transition to the more systematized and formalised structural control concepts advanced in the present work. It should be emphasized that the solution techniques used in control are solely extensions to established mathematical extremisation processes available prior to the advancement of systems concepts. Hence the adoption of control's solution techniques in structural applications appears natural in the evolution of structural optimization. However it is the conceptual framework of control theory which offers the power to the approach, a fact only realized by the references in the following subarticle (§B.1.2). Generally closed-form solutions have only been derived for the simplest of problems. Some form of discretization scheme combined with numerical solution methods appears mandatory for practical problems. But this should not preclude attempts in the future to obtain the far more useful closed-form solutions.

B.1.2 The modelling problem. The schools of thought, as presented independently by Clyde (1970a, 1970b) and Pister (1972), serve as the basis for the present investigation. Clyde has given a systems interpretation to a structure and shown its significance to the framework of the design process. A series of recent communications between Clyde and the author have extended the earlier basis for such an approach. Such work is complemented by Pister's suggested viewpoints on simulation and modelling.

The importance of the conceptual framework suggested by these authors for the modelling of structures is emphasized. The organisation of their thinking of structures has enabled them, in the opinion of the writer, to be more lucid than conventional presentations of structures theory while they have established the basis for the development of an ordered theory of structures. The understanding of structural action is noticeably enhanced. The generality and the systematic thinking of the approaches is appealing.

These few works, although chronologically following the earliest control analogies just cited (§B.1.1), appear to have evolved separately. They generally anticipate the design problem and the consequent control analogies. Historically, considerations have been directed toward single-level models, only Clyde recognising the hierarchical nature of structural systems. The hierarchy has always existed but has lacked definitive treatment.

In view of the important nature of the rational modelling of structures it is surprising to find that it has occupied the time of so few researchers. (See for example the comments of Gregory 1963, Rozvany 1966, Brotchie 1967.) The term 'rational' is used here in the sense of Clyde and Pister to convey a sense of ordering of information and logical reasoning imparted to the modelling process and something that is repeatable between different structures. Researchers have tended to avoid the real problem (namely the rational modelling of structures) in preference for marginally worthwhile activities such as many existing clever applications of optimization techniques. (See §B.1.1.) Real progress in the field of structural mechanics will not be made until the general concepts of modelling have been defined. Systems thinking, in formats similar to Clyde and Pister, appears at the present time to offer distinct advantages in this direction.

B.1.3 Literature pertaining to control theory. Considerable interest has been shown by applied mathematicians and theoretical engineers in control and in particular optimal control in recent years and consequently there exists a large body of literature pertinent to the topic, much at a very high mathematical level. Nevertheless only those publications which most strongly influenced the author or are typical of a viewpoint have been referenced in the text. Where possible the original work has been referenced. Publications giving bibliographical surveys of a particular control topic are noted so as to provide an historical orientation of the topic as well as its perspective in the control systems field generally.

In the formulation of the present modelling and design theory, the following basic references were found to be particularly comprehensive and lucid in their respective fields.—— The surveys of Athans (1966), Butkovskii et al (1968) and Robinson (1971) give very detailed state of the art reports and extensive literature citations. The last two cover distributed parameter systems exclusively where the most definitive works in this field remain Wang and Tung (1964), Wang (1964), and Butkovskii (1969). For optimal control of lumped parameter systems, perhaps the most complete works are Fel'dbaum (1965) for both an overview and solution techniques, and Rozonoer (1959), Pontryagin et al (1962), Lee (1964), Leitmann (1966), Pallu de la Barriere (1967) and Boltyanskii (1971) for solution techniques. Stochastic lumped parameter systems are capably handled by Aoki (1967) and Bellman (1961) who generally use a form of dynamic programming methodology for design problem solutions. Dreyfus (1965) is also a standard reference on dynamic programming, though in a deterministic sense. He relates the techniques of dynamic programming and the calculus of variations in an easily read style. Berkovitz (1961) relates control problems to the calculus of variations. General systems theory (Klir 1969), hierarchical, multilevel systems theory (Mesarovic et al 1970), systems reliability (Gnedenko et al 1969), and systems engineering (Hall 1962) complete the systems disciplines in this thesis. Many additional references of a more specialist nature have also been influential in the formulation of the present work but are cited in the appropriate location of the thesis.

B.2 DETAILED OUTLINE OF THESIS

The remainder of the presentation has been partitioned into four parts containing a total of fourteen sections. The sequence of sections is intended as a logical progression through the thesis. (The stochastic design work of part 3 may however be considered independently of the deterministic design work of part 2 and the system model type I, II and III groupings in part 2 independently of each other.)

Structures are modelled within the framework of a generalised lumped and distributed parameter systems theory. The format is one of treating structures as a whole rather than the fragmented approach of established texts on mechanics. The breadth of the approach is implicit in the use of the term 'system'. The presentation is also systems oriented in that it proceeds from concepts to theory to application. This generalisation permits the system to be specified independently of particular cases which then only require the correct identification of the system elements for definition. The framework of the modelling remains constant for all systems.

Both deterministic and stochastic cases are treated. In control terminology, results relevant to a broad class of structural behaviour are derived, and a specific mathematical formulation of the design problem (single-level, single-goal) is given. The emphasis at all times is on the development of a rational and systematic approach to design and modelling.

The treatment is confined to systems described by vector partial differential equations in the four dimensional space-time domain and in general will be nonlinear in the dependent variables. Discrete modelling will only be discussed in relation to the approximations of this form where it is desirable for solution purposes. This specification encompasses a broad class of (though obviously not all) problems. It is assumed that the system models do not change their properties; that is adaptive and learning systems are excluded. The concepts developed are clarified with illustrations taken from the first order elastic flexural (and shear) theories of beams, plates and shells.

The design of deterministic systems is accomplished through distributed parameter extensions of the maximum principle of Pontryagin. The derivations of these sets of necessary conditions for optimality, through variational arguments after Rozonoer, the dynamic programming technique of Bellman and the approach of classical calculus of variations, for three distinct system models (types I, II and III respectively), are given in detail. The conditions take the form of partial differential equations of the boundary value type. Stochastic design is based on methodical arguments and leads to a recurrence relation for a solution. Markov properties are assumed. Reliability constraints, sensitivity and the singular formulations of design problems are discussed.

The design tools developed for the single-level, single-goal problem are of comparable generality to the single-level modelling and cover a wide spectrum of problems. Their use is illustrated with a devised problem and problems after Armand.

The content of the ensuing sections of the thesis is briefly as follows:

Part 1; Modelling with Reference to the Optimum Problem.

Having introduced a systems approach and developed certain systems ideas in the preceding sections through particular illustrations, the notions are generalised in this part to encompass a broad class of systems and design problems. Part 1 outlines an axiomatic-type presentation of single-level, single-goal modelling according to a control systems theory interpretation. The aim of the modelling ideas adopted is a generalisation and systematization of the modelling and design problems, along with a definition of these problems.

Section §C introduces the necessary notations and terminology in a semi-formal manner and gives an overview of the mathematical modelling issue. The fundamental notions of state and control are given a precise meaning suitable as a foundation for a theory of modelling and design. The system model is then defined in terms of particular mathematical relationships of these entities.

Section §D comprises the core of the mathematical representation of the design problem ingredients. In addition to the system model defined in §C, optimal design requires the further specification of design constraints and design criteria to complete the information necessary in optimal control theory.

Section §E. Having established the system model and components of a design problem, a summary statement of this problem follows and links the information of §D. Approximations, and a preliminary view of and background to solution techniques of the optimum problem follow.

Part 2; Deterministic Design

Part 2 develops the solution techniques - the mathematical tools - required for solving the optimum deterministic single-level, single-goal structural design problem. The formal derivations for the three main structural system models are approached in three complementary ways and lead to equivalent necessary optimality conditions. Their usage is illustrated on a single problem to demonstrate this equivalence. Singular formulations of design problems are discussed.

Section §F derives a set of optimality conditions for the system model type I. Variational arguments are used, following the work of Rozonoer, to reach an extended form of Pontryagin's maximum principle. Extensions to the optimality conditions are detailed to encompass a broad class of problems. The conditions take the form of partial differential equations of the boundary value type and appear as auxiliary equations to be solved simultaneously with the system equations. Boundary conditions for these auxiliary equations appear as additional conditions.

Section §G. The use of these optimality conditions in design is illustrated with reference to an elastic plate problem posed by Armand. The section emphasizes the theoretical problems which may arise and the mathematical procedures for dealing with them. State-control modelling notions are emphasized in the formulation of the problem. A numerical solution is given.

Section §H derives the second set of optimality conditions in part 1, but now for system model type II. The conditions are analogous to those derived in §F for model type I. Dynamic programming techniques are used in the derivation mainly for their ease of application for the system type at hand, but also to give a different view on the derivation of optimality conditions to that employed in §F.

Section §I. Using the same problem of Armand offered in §G, the equations are reinterpreted as a system model type II, again using state-control notions. The necessary conditions derived in §H are then applied. The solution is shown to reduce to the solution of the same equations as those obtained in §G. An alternative equation reduction scheme is outlined leading to a form studied in the fundamental work of Lurie. The solution for this form is again the same. Sensitivity is discussed.

Section §J. A classical variational calculus approach, assuming possible discontinuities in the controls, is utilized to derive the necessary conditions for a stationary value of the design criterion with a type III system model constraint. The Weierstrass-Erdmann corner conditions at the discontinuities also unfold in the derivation. The results are extended in deriving local necessary minimizing conditions. Interpreted in Hamiltonian notation, this leads on to a special statement of a maximum principle for the particular design problem with type III models. The optimality conditions are analogous to those derived in §F and §H.

Section §K. Continuing with the previous plate illustration, the equations are modelled in various ways according to a type III format. One model is shown to coincide in form with that outlined in the foundation studies of A.I. Egorov. The necessary conditions for optimality of the design derived in the previous section (§J) are applied and the resulting solutions are shown to be the same as that obtained in §G and §I for type I and type II system models respectively.

Section §L. In setting up structural design problems, care has to be exercised on the part of the designer when singular formulations are introduced. 'Singular' implies that the optimality conditions of §F, §H and §J may only be applied after suitable modifications and may not be applied directly. The occurrence of the conditions producing singularities is shown in some previous structures work;

by not recognising these conditions the results of this work are not as strong as intended by their authors, namely their solutions are only extremal and not necessarily minimizing. A singular problem is reworked and the results strengthened to illustrate the routine involved in solving singularly formulated problems.

Part 3; Stochastic Design

The experience gained with the deterministic design in part 2 provided the encouragement to extend the treatment to the stochastic case. As implied, the assumption of determinism is put to one side and the alternative assumption of stochasticism is taken up. Conditions are derived and illustrated for the optimum design of mathematical models that allow uncertainty in a probabilistic sense. Markov properties are assumed. Reliability constraints are included.

Section §M derives conditions that the system model must satisfy in order for it to be optimal. Methodical arguments are employed in the derivation although it is shown that similar results could be obtained by applying Bellman's principle of optimality directly. The results are applicable for two alternative formulations where the probability laws or first and second moments are specified for the structural system's variables. In either case, the conditions assume the form of a recurrence relation.

Section §N shows that with little modification the conditions of §M are applicable where reliability constraints are present in the design brief. Reliability is used in the same sense as that describing the system, namely learning or adaptive systems are excluded - the system properties are assumed to be unchanging. As a result of incorporating reliability, the dimensionality of the problem rises which in certain cases may make the design problem difficult to manage.

Section §O illustrates the use of the conditions of §M and the stochastic modelling of part 1 in a state-control sense. A numerical solution is given to the example.

Part 4; Closure, References and Appendices

Section §P. Conclusions and discussion regarding the proposed modelling techniques and means of structural design are outlined in this last section. Suggestions for further activity in the field and the direction in which the subject may head according to the writer's opinion may also be found here.

References are arranged in alphabetical/publication date sequence.

Appendix one delineates two algorithms for the reduction of a general high order partial differential equation to the standard form of system model types I and III without regard to the meaning of the resulting variables or equations. The reductions are complementary to those outlined in §C.

Appendix two defines the relevant reliability analysis problem in a formulation consistent with the state-control concepts of system modelling adopted in this thesis. As the wording 'analysis' implies, the work is of a different character to much of the detail in the main body of the thesis. It is shown that state notions are a natural means of describing reliability. The reliability results are used in section §N.

Appendix three gives an explicit form for a one-parameter family of surfaces in a spherical polar coordinate system. A one-parameter family of surfaces is used in a qualitative sense in §H.

PART 1

MODELLING WITH REFERENCE TO THE OPTIMUM PROBLEM

§C SYSTEM MODEL CHARACTERISTICS

C.1 GENERAL.

The use of state and control variables provides a measure of generalisation or uniformity to the composition of structural models. The inter-relationship of these variables is expressed through the constitutive equation. Under suitable transformations, with a mechanics basis, the constitutive equation may be reinterpreted in general state equation forms.

Section §A gave the fundamental ideas of state and control in a relatively informal manner. It stressed the choice of the variables from an engineering and physical motivational viewpoint. These basic ideas are extended here in generality and abstraction to model a wide spectrum of structures. Geometrical interpretations are attempted throughout, often by the use of representative spaces to describe the system. Illustrations are used to enable the presentation of the theory to be of a reasonably formal nature. At the level of generality given, the underlying principles and assumptions of mechanics come more to the fore as it requires a clarification of structural thinking in order to formulate the models.

Article §C.2 introduces the necessary notations and terminology by giving precise meanings to the notions of state, control and response. The system model is then defined in article §C.3 in terms of particular mathematical relationships of these entities. In general, following the developments of section §A, the constituent parts of the system model will consist of (i) the state equations - §C.3.1, (ii) certain 'starting' information on the states - §C.3.2, and (iii) the state-response description - §C.3.3.

C.2 PRELIMINARY DEFINITIONS, PROPERTIES AND ASSUMPTIONS.

Consider a spatial domain D, with boundary ∂D, defined in three dimensional Euclidean space E^3, with spatial coordinate vector $\underline{y} = (y_1, y_2, y_3)^T$. Let the time domain be T, an interval $[t^L, t^R]$ of the real line E^1.

The state of the model, defined for all $\underline{y} \in D$, at any time instant $t \in T$, is denoted by the n-valued vector function $\underline{x}(t, \underline{y})$ with coordinates

$\{x_i(t, \underline{y});\ i = 1, \ldots, n\}$. For given values of t and $\underline{y}$, the x_i may be conveniently regarded as the variables characterizing the internal behaviour or state of the system model. They are 'controlled' variables. For example, in a beam model, with static flexural deformations only considered, the state is specified by four variables - the deflection and its three successive derivatives (related to slope and the notions of moment and shear force respectively). In certain circumstances it is convenient to think of the set of all possible functions $\{\underline{x}(t, \underline{y});\ t \in T,\ \underline{y} \in D\}$ defined on $T \times D$ (the product space of T and D) as comprising the 'state function space', in which the x_i are coordinates. Other state spaces may equally well be defined (such as for example are discussed in Balakrishnan 1965, Greenberg 1971).

<u>Control</u> is maintained on the model by the r-valued control vector function $\underline{u}(t, \underline{y})$ with components $\{u_j(t, \underline{y});\ j = 1, \ldots, r\}$. u_j may be defined over all or part of the spatial domain D. For the same example, the control is provided by the choice of beam rigidity. With a similar intent for the introduction of the state function space defined above, the set of all possible functions $\{\underline{u}(t, \underline{y});\ t \in T,\ \underline{y} \in D\}$ defined on $T \times D$ may be thought of as a control function space.

The <u>response</u>, or outward behaviour variables $\underline{z}(t, \underline{y})$ with components $\{z_k(t, \underline{y});\ k = 1, \ldots, m \leq n\}$ will be related to the state by suitable algebraic transformations. For the specific structural example at hand, deflection may be considered as the response and this is also the first state coordinate. In this case the relationship of state and output is a linear transformation of the general form

$$z_k(t, \underline{y}) = \sum_{i=1}^{n} a_{ki}\, x_i(t, \underline{y}) \qquad k \leq n$$

where a_{ki} are constants.

As defined, $\underline{u}$ and $\underline{x}$ (and $\underline{z}$) are continuous functions in level (that is in control and state function spaces respectively) and over the parameter sets t and $\underline{y}$. The systems to be considered will in general be nonlinear in both the state and control. In a sense, the control (along with known end-state conditions - see §A.2) forms an input to the model, while the response is an output form. The state represents the internal behaviour of the model; for structural models a transformation is always known for any given model between state and response and knowledge of one defines the other.

In subsequent discussions, for the range and behaviour of structural models treated, the literal distinction between the time domain (T) and the space domain (D) will be eliminated for mathematical convenience although it is acknowledged that they are essentially different physically. To enable this, the parameter vector $\underline{y}$ will be enlarged to contain in general four components $\{y_i; i = 1, \ldots, 4\}$ where it is recognised that $y_4 \hat{=} t$. Only during particular applications or illustrations will reference be made to time or space explicitly. Such reasoning allows a discussion of static and dynamic structural behaviour to proceed in the same format. The resulting four dimensional Euclidean space of the parameters $\{y_i; i = 1, \ldots, 4\}$ will be denoted Y, $(Y \hat{=} T \times D)$, and the intervals of variation of y_i by $[y_i^L, y_i^R]$; that is $y_i \in [y_i^L, y_i^R]$, $i = 1, \ldots, 4$. (The superscript L, R notation implies a visualization of the coordinate space where, loosely, L stands for the 'left-hand' and R the 'right-hand' end points of this space.)

The above descriptions apply to <u>deterministic models</u>, that is models with properties which are considered to be of known value and fully predictable. An alternative system model, the <u>stochastic</u> or <u>probabilistic model</u>, regards the model properties as random with given probability distributions (and hence not uniquely definable in advance) and which are used to calculate the probable response of the model; the behaviour is given to lie within certain probabilistic bounds. (The terms stochastic, probabilistic and random are used interchangeably.) Provided the probability distributions are specified in advance there are no prominent dissimilarities between deterministic and stochastic treatments. The above deterministic-stochastic classification of models is the one commonly accepted in systems theories. (See for example Bellman 1961, Tsypkin 1971.)

<u>A random field assumption:</u> To extend the concepts of state and control to a probabilistic format, the state and control functions are endowed with an additional parameter set Ω (a sample space) such that $\{\underline{x}(\omega, \underline{y}); \underline{y} \in Y, \omega \in \Omega\}$ now represents a 'random field'. The control and output functions are defined equivalently. A random field is here

taken to denote a family of random variables indexed by points in the parameter space $Y \subset E^4$ (although this space may correspond to an abstract space in the general theory of random fields - See Wong 1971.)

A random field may be regarded as the multidimensional equivalent of a stochastic process (random process or random function). A stochastic process is indexed by a single parameter, y say. (See for example Doob 1953, Loeve 1963.) For each y, $\underline{x}(\omega,\cdot)$ is defined on a sample space Ω and is a 'random vector'. For each probability parameter $\omega \in \Omega$, $\underline{x}(\cdot,\ y)$ is a function of y and is termed a 'realization' ('sample function') of the process.

It is assumed that the statistical characteristics of the random vectors (variables) are known. That is the system is stochastic in the sense just defined. The most suitable way to characterize random vectors (variables) is by the notation $\underline{x}(y)$ where the probability parameter is understood. It will generally be clear from the context whether deterministic or stochastic concepts are implied.

Notice that the deterministic state-modelling notions extend readily to the probabilistic case by endowing the state, control, output and model parameters with random field properties. Deterministic differential (difference) equations will be seen to have equivalent probabilistic counterparts, namely stochastic differential (difference) equations.

C.3 SYSTEM MODELS.

C.3.1 The state equations.

C.3.1.1 Introduction. The state equations are an alternative expression of the constitutive relationship and show the dependence of the state on the control (in addition to other system model parameters). In the following this dependence will be taken to be in the form of time and space differential operators and in general will be nonlinear. The choice of the constitutive relationship is based on the designer's comprehension of the structural action involved; (for example shell action, plate action - that is whether the designer considers the system will behave as a shell, plate or other structure.)

The model will be defined in the four dimensional space Y, or subspaces of Y, and expressed in state form. With the notes and assumptions of the foregoing article (§C.2), the state equations will assume the form of vector partial differential equations defined over a space-time domain. Three standard forms are considered and for purposes of distinction will be arbitrarily labelled types I, II and III (§C.3.1.2, §C.3.1.3 and §C.3.1.4 respectively). They are the only standard distributed parameter forms that appear in the control literature where their occurrence is typically a specialized form of versions given here.

The choice and usage of these three forms in the present structures case requires explanation. Commencing with type III, it is emphasized that this type has only been included in the present treatment for completeness because, as mentioned above, it occurs in the control literature. It is shown that structures can be interpreted in this form, and its usage on occasions may be favourable computationally, but the form is not favoured by the writer as it lacks complete physical interpretation. The remaining discussion in the following two paragraphs on the choice of the state equation forms will therefore wholey centre on types I and II.

In deciding on the form of state equations, standard forms are sought and in particular standard first order forms are sought. The first order requirement foresees possible numerical solutions of the differential equations, solutions which are usually only valid for first order equations. The first order nature also follows from the notion of state; the state vector components for structures will invariably be related to the derivative of the adjacent state components. Thus when the state components are differentiated this will lead to an ordered hierarchy of first order equations. Lastly the first order form allows any general n'th order equation to be reduced to a standard form, allowing a common description of all systems.

Thus two fundamental first order forms arise (with, obviously, intermediate combinations possible). In particular state equations type I express the behaviour of the model in terms of the behaviour in the direction of a single independent parameter. State equations type II give equal emphasis to all the independent parameters. The two forms may be considered as alternatives for different modelling situations, the choice of usage being left to the applier. The equation types apply to any domain dimension equal to or below that described in §C.3.1.2 and §C.3.1.3 for the respective type; their use is not restricted to any particular domain dimension within this range. Type III is specifically for a two dimensional domain although it could be extended to a three dimensional domain as noted in §C.3.1.4. The equation type gives equal emphasis to both independent parameters but in a different sense to the two dimensional form of type II. It is remarked in passing that various combinations of types I, II and III forms could conceivably be constructed, but their use would not be favoured owing to their lack of physical interpretation.

C.3.1.2 State equations type I. (see for example Wang and Tung 1964, Sirazetdinov 1964)

$$\text{(c.3.1)} \qquad \frac{\partial \underline{x}(\underline{y})}{\partial y_4} = \underline{f}[\underline{y},\ \underline{x}(\underline{y}),\ \dots\ ,\ \partial_{\underline{\ell}}\underline{x}(\underline{y}),\ \dots\ ,\ \underline{u}(\underline{y})]$$

where $\underline{y} = (y_1,\ \dots\ ,\ y_4)^T$, $\underline{\ell} = (\ell_1,\ \ell_2,\ \ell_3)$ and $\partial_{\underline{\ell}}\underline{x}$ is as defined in the 'notation'. $\underline{f}$ is defined over the domain $Y \subset E^4$; $\underline{f} = (f_1,\ \dots\ ,\ f_n)^T$ and in general is a nonlinear vector-valued function of both the equation dependent variables $\underline{x}$ and $\underline{u}$ and the equation independent variables $\underline{y}$.

The choice of derivatives with respect to y_4 appearing on the left hand side is arbitrary. In fact derivatives with respect to any y_i, $i = 1, 2, 3, 4$, are permissible on the left hand side provided derivatives with respect to the same parameter do not occur on the right hand side.

Equation (c.3.1) is an extension of the finite dimensional standard equation (a.2.4). (By the introduction of additional state variables for the state derivatives on the right hand side of (c.3.1), equation (c.3.1) could be further reduced to a set of first order equations - see for example Courant and Hilbert 1962 - but then this extension would not apply.)

C.3.1.3 State equations type II. (see for example Lurie 1963, Butkovskii et al 1968)

$$\frac{\partial \underline{x}(\underline{y})}{\partial y_i} = \underline{f}^i[\underline{y}, \underline{x}(\underline{y}), \dots, \partial_{\underline{\ell}}\underline{x}(\underline{y}), \dots, \underline{u}(\underline{y})] \qquad (i = 1, 2, 3) \tag{c.3.2}$$

where $\underline{\ell} = (\ell_h, \ell_k)$, $(h, k = 1, 2, 3;\ h, k \neq i)$ and $\partial_{\underline{\ell}}\underline{x}$ is as defined in the 'notation'. $\underline{f}^i$, $i = 1, 2, 3$, are in general nonlinear vector-valued functions of the arguments shown and have to be such that they satisfy certain compatibility conditions (Ames 1965); that is every solution of one equation is a solution of the other equations.

Equations (c.3.2) are a special form of the Pfaffian system of equations (Lurie 1963, Haack and Wendland 1972). The vector $\underline{y}$ is taken to have only three components (y_1, y_2, y_3) here (compare with four for type I) as the model is primarily intended for system models described over a three-dimensional spatial domain.

In (c.3.2) the derivatives of state with respect to each of the independent variables y_i have been isolated on the left hand side. Derivatives of state with respect to the remaining independent variables y_h, y_k $(h, k \neq i)$ may occur on the right hand side. As for the type I equations, (c.3.2) reduce to the standard lumped parameter form (equation a.2.4) as a special case when $\underline{y}$ is one dimensional.

<u>C.3.1.4 State equations type III.</u> (see for example A.I. Egorov 1963, 1964, Butkovskii 1969)

$$\frac{\partial^2 \underline{x}(y)}{\partial y_1 \partial y_2} = \underline{f}[\underline{y}, \underline{x}(\underline{y}), \dots, \partial_{\ell}\underline{x}(\underline{y}), \dots, \underline{u}(\underline{y})] \tag{c.3.3}$$

where $\underline{y} = (y_1, y_2)^T$, ℓ is both ℓ_1 and ℓ_2 but never ℓ_1 and ℓ_2 together in the same term, $\partial_{\ell}\underline{x}$ is as defined in the 'notation', and $\underline{f}$ is in general a nonlinear vector-valued function of the arguments shown.

The equations are applicable in descriptions over two dimensional planar regions. The left hand sides are now (compare with types I and II) second order derivatives corresponding to the isolation of the mixed derivative terms from the remaining derivative terms. No mixed derivatives of state appear on the right hand sides. The form, unlike (c.3.1) and (c.3.2), is not reducible to the lumped parameter version (equation a.2.4) in transferring to one dimensional $\underline{y}$, essentially because (c.3.3) highlights the mixed derivative terms which are absent in the lower dimension. There is a more general case than (c.3.3), namely that which applies over a three dimensional domain (mixed third order derivatives occur on the left hand sides), but its usefulness is doubtful essentially because of the occurrence of the higher derivatives.

C.3.1.5 A note on the controls. In equations (c.3.1, 2, 3) the control appears only on the right hand sides and is independent of differential operators. Both requirements are for convenience in controlling the system. The second requirement especially prevents sudden changes in the control (for example abrupt changes in the geometry of the structural member) from causing jumps in the values of one or more of the state variables. In the form of (c.3.1, 2, 3), the states may not change by finite amounts at a given location unless the control contains the equivalent form of an impulse (which clearly does not occur in structures). It may also be seen that if derivatives of control appear in the state equations, there exists an ambiguity in the design problem. In particular at each location not only has a control to be chosen but also the derivative(s) of the control, the latter obviously defines the control at an adjacent location and hence the control at this last location cannot be freely chosen.

C.3.1.6 Methods of reduction (decomposition). The form of equations (c.3.1, 2, 3) is consistent with the reduction of a high order, with respect to y_i (i = 4 for (c.3.1); i = 1, 2, 3 for (c.3.2); i = 1, 2 for (c.3.3)), partial differential equation to a set of first order (second order for III) differential equations. These are the so called state variable or state space formulations of the system equations. Conventional structural thinking would tend to operate in the reverse manner to this, namely to eliminate all but one of the behaviour variables (the response variable), yielding a single high order equation governing the behaviour of the model. Any partial differential equation may be reduced to (c.3.1, 2, or 3) by introducing suitable (state) variables.

In the general mathematical theory of control, without regard to the physical meaning of the variables involved, suitable reduction or decomposition procedures are well established for ordinary differential equations of order greater than one. (See for example Pontryagin 1962.) The extension to partial differential equations is slightly more complicated owing to the additional cross derivatives with respect to the several independent variables. (See Courant and Hilbert 1962.) As with the ordinary differential equation case, the many and various forms of possible decompositions suggest a nonunique property of the decompositions and the availability of various choices as a basis for the decomposition. Any set of state and control variables may be chosen provided the set satisfies equation (c.3.1) (or (c.3.2) or (c.3.3) if these equation types are sought). This degree of freedom in choice of the method of decomposition as well as in the choice of equation form (that is type I, II or III), however, does not vary the final result of manipulations of the equations. This may be demonstrated, for example, in the problem of optimization - see the illustrations in sections §G, §I and §K or Armand (1972).

Armand (1972) in his optimization studies, following the work of Lurie (1963), adopts a particular form of type II (namely two dimensional and no derivatives of state appearing on the right hand side). The particular form of equation (c.3.2) has generally (part B of Armand 1972 is an exception - see later comments) been satisfied in a mathematical sense with the state and control variables often lacking physical significance. Decomposition algorithms leading to sets of equations of the form (c.3.1) and (c.3.3) have been formulated by the writer in correspondence with Lurie's and Armand's work on equations of the form (c.3.2). Since equivalent decomposition techniques cannot be found in the literature, they are included in appendix one. The type I decomposition is an extension of the decomposition algorithm of Wang and Tung (1964). As with the work of Lurie and Armand, the variables generally lack physical meaning; in particular, geometry terms are chosen as state variables and the controls are chosen as derivatives of the geometry.

If such decomposition algorithms were the only ones possible, the use of control theory in modelling structures would undoubtedly be restrictive. Fortunately this is not the case and decomposition procedures (logically based on the principles of mechanics) can be found which have variables and state equations of physical significance. These are illustrated now.

C.3.1.7. Illustration of the proposed decomposition procedures.

To emphasize the type of decomposition proposed consider as an illustration the shallow shell equations. For zero Poisson's ratio, the equations read (Flugge 1973)

$$\frac{\partial}{\partial y_1}\left[D\left(\frac{\partial w_1}{\partial y_1} - w_3 z_{11}\right)\right] + \frac{\partial}{\partial y_2}\left[\frac{D}{2}\left(\frac{\partial w_1}{\partial y_2} + \frac{\partial w_2}{\partial y_1} - 2w_3 z_{12}\right)\right] + q_1 = 0$$

$$\frac{\partial}{\partial y_2}\left[D\left(\frac{\partial w_2}{\partial y_2} - w_3 z_{22}\right)\right] + \frac{\partial}{\partial y_1}\left[\frac{D}{2}\left(\frac{\partial w_1}{\partial y_2} + \frac{\partial w_2}{\partial y_1} - 2w_3 z_{12}\right)\right] + q_2 = 0$$

$$(c.3.4) \qquad z_{11}\left[D\left(\frac{\partial w_1}{\partial y_1} - w_3 z_{11}\right)\right] + 2z_{12}\left[\frac{D}{2}\left(\frac{\partial w_1}{\partial y_2} + \frac{\partial w_2}{\partial y_1} - 2w_3 z_{12}\right)\right]$$

$$+ z_{22}\left[D\left(\frac{\partial w_2}{\partial y_2} - w_3 z_{22}\right)\right] + \frac{\partial^2}{\partial y_1^2}\left[-K\frac{\partial^2 w_3}{\partial y_1^2}\right] + 2\frac{\partial^2}{\partial y_1 \partial y_2}\left[-K\frac{\partial^2 w_3}{\partial y_1 \partial y_2}\right]$$

$$+ \frac{\partial^2}{\partial y_2^2}\left[-K\frac{\partial^2 w_3}{\partial y_2^2}\right] + q_3 = 0$$

where y_1, y_2, y_3 denote the coordinate axes (cartesian). The middle surface of the shell is described by

$$z = z(y_1, y_2)$$

Derivatives of z with respect to the coordinates y_1 and y_2 are denoted by a subscript notation on z. (For example $z_{12} \triangleq \frac{\partial^2 z}{\partial y_1 \partial y_2}$.)

q_1, q_2, q_3 denote the components of the distributed surface load.
w_1, w_2, w_3 denote the shell deformations.
D and K are the extensional and bending stiffnesses of the shell.

It is remarked that intuitive arguments, similar to that outlined in §A.2, could be employed to give the same decomposition results as detailed here. The following discussion however, attempts to remove much of the qualitative nature of intuitive arguments by converting the decomposition procedures into semi-mechanical routines.

Consider the decomposition process involved in attaining equations of the form type I where the variables will be chosen to take on a physical meaning. The state and control variables chosen will first be listed along with the associated state equations and then the process used to obtain the state, control and state equations will be explained.

The state variables are chosen as

$$
\begin{aligned}
&x_1 \mathrel{\hat{=}} w_1 && x_2 \mathrel{\hat{=}} D\left(\frac{\partial w_1}{\partial y_1} - w_3 z_{11}\right) \\
&x_3 \mathrel{\hat{=}} w_2 && x_4 \mathrel{\hat{=}} \frac{D}{2}\left(\frac{\partial w_1}{\partial y_2} + \frac{\partial w_2}{\partial y_1} - 2w_3 z_{12}\right) \\
&x_5 \mathrel{\hat{=}} w_3 && x_6 \mathrel{\hat{=}} \frac{\partial w_3}{\partial y_1} \qquad\qquad x_7 \mathrel{\hat{=}} K\,\frac{\partial^2 w_3}{\partial y_1^2} \\
&x_8 \mathrel{\hat{=}} \frac{\partial}{\partial y_1}\left(K\,\frac{\partial^2 w_3}{\partial y_1^2}\right) + 2\,\frac{\partial}{\partial y_2}\left(K\,\frac{\partial^2 w_3}{\partial y_1 \partial y_2}\right)
\end{aligned}
\tag{c.3.5}
$$

and the controls $u_1 \mathrel{\hat{=}} D$, $u_2 \mathrel{\hat{=}} K$ (which are functionally related).

Neglecting sign conventions, x_1, x_3 and x_5 are identified as the deformations, x_2 as the in-plane normal force in the y_1 direction, x_4 as the in-plane shearing force, x_6, x_7 and x_8 as the slope, internal moment and 'transverse force' in the y_1 direction. (In fact x_8 is the familiar Kirchhoff combination of twisting moment and out-of-plane shearing force - see for example Timoshenko and Woinowski-Krieger 1959 , Flugge 1973.)

Differentiating the state vector $\underline{x} = (x_1, \ldots , x_8)^T$ with respect to y_1 yields the state equations

$$
\left[\begin{matrix} \dfrac{\partial x_1}{\partial y_1} \\[2ex] \dfrac{\partial x_2}{\partial y_1} \end{matrix}\right.
\qquad
\left[\begin{matrix} \dfrac{x_2}{u_1} + x_5 z_{11} \\[2ex] - q_1 - \dfrac{\partial x_4}{\partial y_2} \end{matrix}\right.
$$

$$(c.3.6)\qquad \begin{bmatrix} \dfrac{\partial x_3}{\partial y_1} \\[2ex] \dfrac{\partial x_4}{\partial y_1} \\[2ex] \dfrac{\partial x_5}{\partial y_1} \\[2ex] \dfrac{\partial x_6}{\partial y_1} \\[2ex] \dfrac{\partial x_7}{\partial y_1} \\[2ex] \dfrac{\partial x_8}{\partial y_1} \\[2ex] \\ \end{bmatrix} = \begin{bmatrix} \dfrac{2x_4}{u_1} + 2x_5 z_{12} - \dfrac{\partial x_1}{\partial y_2} \\[2ex] - q_2 - \dfrac{\partial}{\partial y_2}\left[u_1\left\{\dfrac{\partial x_3}{\partial y_2} - x_5 z_{22}\right\}\right] \\[2ex] x_6 \\[2ex] \dfrac{x_7}{u_2} \\[2ex] x_8 - 2\dfrac{\partial}{\partial y_2}\left\{u_2 \dfrac{\partial x_6}{\partial y_2}\right\} \\[2ex] z_{11}x_2 + 2z_{12}x_4 + z_{22}\left[u_1\left\{\dfrac{\partial x_3}{\partial y_2} - x_5 z_{22}\right\}\right] \\[2ex] + \dfrac{\partial^2}{\partial y_2^2}\left[- u_2 \dfrac{\partial^2 x_5}{\partial y_2^2}\right] + q_3 \end{bmatrix}$$

which for constant or finite dimensional controls are now in the standard form of type I.

The process of obtaining (c.3.5) and (c.3.6) followed the following generalized outline. (The outline will be seen to be a generalisation to distributed parameter system models type I of the lumped parameter example given in §A.2.)

(i) The dependent variables in the original equations may be categorized as 'behaviour' variables and 'geometry' variables. The order of the original equations in the behaviour variables (irrespective of the order in the geometry variables) determines the number of states; for equations of total order n in the behaviour variables (with respect to some independent variable y_a), there result n state variables. The choice of the independent variable y_a is arbitrary.
For the particular example, equations (c.3.4) are second order in w_1 and w_2 and fourth order in w_3 giving a total order of eight (with respect to either

y_1 or y_2). Eight state variables result. For definiteness in this example y_1 has been chosen as the independent variable. The equations (c.3.4) are however symmetrical with respect to both y_1 and y_2, and either could have been chosen. (In certain circumstances it will nevertheless be more favourable to choose a particular independent variable.)

(ii) The n state variables are chosen such that each is related to a b'th derivative (with respect to y_a) of the behaviour variables. For q equations of total order n in behaviour variables, $b = 0,1, \ldots, n_1-1; \ldots ; b = 0, 1, \ldots, n_j-1; \ldots ; b = 0, 1, \ldots, n_q-1$; where $\sum_1^q n_j = n$. That is, for each equation of order n_j in the behaviour variable, n_j states are introduced related to the 0'th order through (n_j-1)'th order of this behaviour variable. The increasing order of the derivatives from 0 to n_j-1 ensures that when the state variables are differentiated with respect to y_a, the first order state equations (total order n_j) are equivalent to the original n_j'th order equation.
For example, consider equation $(c.3.4)^3$. (Equations $(c.3.4)^1$ and $(c.3.4)^2$ follow similar arguments; the resulting states and state equations are additive to those obtained from $(c.3.4)^3$.) The states $x_5, \ldots, x_8$ where chosen related to the b'th derivative of w_3 (total order four with respect to y_1 in equation $(c.3.4)^3$) where b ranged from 0 to 3.

(iii) The detail of the state variables is adjusted to coincide with the definition of a meaningful quantity, while the resulting state equations should not only conform to the type I format but should also be equivalent to the original n_j'th order equation. The state equations should also be able to be interpreted in terms of equilibrium, compatibility and constitution at what would be the lower level of an equivalent multilevel model.
Continuing with the example of equation $(c.3.4)^3$, the states $x_5, \ldots, x_8$ were adjusted in detail to take on the meanings of deflection, slope, internal moment and Kirchhoff 'transverse force' respectively, while the resulting state equations may be given the following interpretation: $(c.3.6)^5$ and $(c.3.6)^6$ represent compatibility and constitution combined, and $(c.3.6)^7$ and $(c.3.6)^8$ together represent equilibrium. Equations $(c.3.6)^{5\to 8}$ together are equivalent to the original fourth order equation $(c.3.4)^3$.

(iv) Control variables choose themselves. They represent the physical properties of the structure.
In the present example, the controls were chosen as the extensional and bending stiffnesses.

To complete the discussion on equations (c.3.6), for the two in-plane portions, equilibrium is represented by the second and fourth equations while the conditions of compatibility and constitution combined are contained in the first and third equations.

Similar decompositions may be performed to yield equations type II. To illustrate the decomposition procedure involved and as a comparison with the decomposition just given for type I, consider the shell equations once more. Again it will prove convenient to list the states, controls and state equations and then to outline the process whereby these were obtained. The same notation $\underline{x}$ and $\underline{u}$ will be used for the state and control again although equivalence with (c.3.5) is not implied.

The following state variables are introduced

$$x_1 \triangleq w_1 \qquad x_2 \triangleq w_2$$

$$x_3 \triangleq D\left(\frac{\partial w_1}{\partial y_1} - w_3 z_{11}\right) \qquad x_4 \triangleq \frac{D}{2}\left(\frac{\partial w_1}{\partial y_2} + \frac{\partial w_2}{\partial y_1} - 2w_3 z_{12}\right)$$

$$x_5 \triangleq D\left(\frac{\partial w_2}{\partial y_2} - w_3 z_{22}\right)$$

$$x_6 \triangleq w_3 \qquad x_7 \triangleq \frac{\partial w_3}{\partial y_1} \qquad x_8 \triangleq \frac{\partial w_3}{\partial y_2}$$

(c.3.7) $$x_9 \triangleq K\frac{\partial^2 w_3}{\partial y_1^2} \qquad x_{10} \triangleq K\frac{\partial^2 w_3}{\partial y_1 \partial y_2} \qquad x_{11} \triangleq K\frac{\partial^2 w_3}{\partial y_2^2}$$

$$x_{12} \triangleq \frac{\partial}{\partial y_2}\left[K\frac{\partial^2 w_3}{\partial y_1^2}\right] \qquad x_{15} \triangleq \frac{\partial}{\partial y_1}\left[K\frac{\partial^2 w_3}{\partial y_2^2}\right]$$

$$x_{13} \triangleq \frac{\partial}{\partial y_1}\left[K\frac{\partial^2 w_3}{\partial y_1^2}\right] + \frac{\partial}{\partial y_2}\left[K\frac{\partial^2 w_3}{\partial y_1 \partial y_2}\right]$$

$$x_{14} \triangleq \frac{\partial}{\partial y_2}\left[K\frac{\partial^2 w_3}{\partial y_2^2}\right] + \frac{\partial}{\partial y_1}\left[K\frac{\partial^2 w_3}{\partial y_1 \partial y_2}\right]$$

The controls $u_1 \triangleq D$, $u_2 \triangleq K$ remain the same. Notice that for each derivative of w_i, $i = 1, 2, 3$ to the order α, there were introduced $\alpha+1$ state variables, so that a pyramid effect results with (for example considering w_3) the apex at $\alpha=0$ (one state variable, namely x_6) and base at $\alpha=3$ (four state variables, namely x_{12} , ... , x_{15}). The interpretation of the state variables is evident; in particular for the in-plane equations they are displacements, normal and shearing forces; for the out-of-plane equation they are displacement, slopes, internal direct and twisting moments, and internal shearing forces, all in the y_1 and y_2 directions where applicable. In addition two coupling state variables x_{12} and x_{15} appear. They have no accepted appellation but may be given physical meaning.

To ensure equation equivalence when the state variables are differentiated, certain auxiliary dependent variables are required. They may be treated as auxiliary controls although they may not be directly altered by the designer in the manner u_1 and u_2 can be. (See Lurie 1963, Armand 1972.) Set

$$u_3 \triangleq \frac{\partial}{\partial y_1}\left[D\left(\frac{\partial w_2}{\partial y_2} - w_3 z_{22}\right)\right] \qquad u_4 \triangleq \frac{\partial}{\partial y_2}\left[D\left(\frac{\partial w_1}{\partial y_1} - w_3 z_{11}\right)\right]$$

$$u_5 \triangleq \frac{\partial}{\partial y_1}\left\{\frac{\partial}{\partial y_2}\left[K\frac{\partial^2 w_3}{\partial y_1^2}\right]\right\} \qquad u_6 \triangleq \frac{\partial}{\partial y_2}\left\{\frac{\partial}{\partial y_2}\left[K\frac{\partial^2 w_3}{\partial y_1^2}\right]\right\}$$

$$u_7 \triangleq \frac{\partial}{\partial y_1}\left\{\frac{\partial}{\partial y_1}\left[K\frac{\partial^2 w_3}{\partial y_1^2}\right] + \frac{\partial}{\partial y_2}\left[K\frac{\partial^2 w_3}{\partial y_1 \partial y_2}\right]\right\}$$

(c.3.7)'

$$u_8 \triangleq \frac{\partial}{\partial y_2}\left\{\frac{\partial}{\partial y_1}\left[K\frac{\partial^2 w_3}{\partial y_1^2}\right] + \frac{\partial}{\partial y_2}\left[K\frac{\partial^2 w_3}{\partial y_1 \partial y_2}\right]\right\}$$

$$u_9 \triangleq \frac{\partial}{\partial y_1}\left\{\frac{\partial}{\partial y_2}\left[K\frac{\partial^2 w_3}{\partial y_1^2}\right] + \frac{\partial}{\partial y_1}\left[K\frac{\partial^2 w_3}{\partial y_1 \partial y_2}\right]\right\}$$

$$u_{10} \triangleq \frac{\partial}{\partial y_1}\left\{\frac{\partial}{\partial y_1}\left[K\frac{\partial^2 w_3}{\partial y_2^2}\right]\right\} \qquad u_{11} \triangleq \frac{\partial}{\partial y_2}\left\{\frac{\partial}{\partial y_1}\left[K\frac{\partial^2 w_3}{\partial y_2^2}\right]\right\}$$

During the optimization procedures to be outlined in part 2, the equations involving the auxiliary control variables disappear and hence are not carried through the computations. Auxiliary control variables were implicit in the modelling required for equations type I but were never employed as equivalence of the decomposed equations and the original high order equation was satisfied trivially.

Differentiating the state variables with respect to y_1 and y_2 in turn, a set of first order equations (the state equations) is obtained equivalent to the original shell equations

$$\frac{\partial x_1}{\partial y_1} = \frac{x_3}{u_1} + x_6 z_{11} \qquad \frac{\partial x_1}{\partial y_2} = \frac{2x_4}{u_1} - \frac{\partial x_2}{\partial y_1} + 2x_6 z_{12}$$

$$\frac{\partial x_2}{\partial y_1} = \frac{2x_4}{u_1} - \frac{\partial x_1}{\partial y_2} + 2x_6 z_{12} \qquad \frac{\partial x_2}{\partial y_2} = \frac{x_5}{u_1} + x_6 z_{22}$$

$$\frac{\partial x_3}{\partial y_1} = - q_1 - \frac{\partial x_4}{\partial y_2} \qquad \frac{\partial x_3}{\partial y_2} = u_4$$

$$\frac{\partial x_4}{\partial y_1} = - q_2 - \frac{\partial x_5}{\partial y_2} \qquad \frac{\partial x_4}{\partial y_2} = - q_1 - \frac{\partial x_3}{\partial y_1}$$

$$\frac{\partial x_5}{\partial y_1} = u_3 \qquad \frac{\partial x_5}{\partial y_2} = - q_2 - \frac{\partial x_4}{\partial y_1}$$

(c.3.8) $$\frac{\partial x_6}{\partial y_1} = x_7 \qquad \frac{\partial x_6}{\partial y_2} = x_8$$

$$\frac{\partial x_7}{\partial y_1} = \frac{x_9}{u_2} \qquad \frac{\partial x_7}{\partial y_2} = \frac{x_{10}}{u_2}$$

$$\frac{\partial x_8}{\partial y_1} = \frac{x_{10}}{u_2} \qquad \frac{\partial x_8}{\partial y_2} = \frac{x_{11}}{u_2}$$

$$\frac{\partial x_9}{\partial y_1} = x_{13} - \frac{\partial x_{10}}{\partial y_2} \qquad \frac{\partial x_9}{\partial y_2} = x_{12}$$

$$\frac{\partial x_{10}}{\partial y_1} = x_{14} - \frac{\partial x_{11}}{\partial y_2} \qquad \frac{\partial x_{10}}{\partial y_2} = x_{13} - \frac{\partial x_9}{\partial y_1}$$

$$\frac{\partial x_{11}}{\partial y_1} = x_{15} \qquad \frac{\partial x_{11}}{\partial y_2} = x_{14} - \frac{\partial x_{10}}{\partial y_1}$$

$$\frac{\partial x_{12}}{\partial y_1} = u_5 \qquad \frac{\partial x_{12}}{\partial y_2} = u_6$$

$$\frac{\partial x_{13}}{\partial y_1} = u_7 \qquad \frac{\partial x_{13}}{\partial y_2} = u_8$$

$$\frac{\partial x_{14}}{\partial y_1} = u_9 \qquad \frac{\partial x_{14}}{\partial y_2} = z_{11}x_3 + 2z_{12}x_4 + z_{22}x_5 + q_3 - u_7$$

$$\frac{\partial x_{15}}{\partial y_1} = u_{10} \qquad \frac{\partial x_{15}}{\partial y_2} = u_{11}$$

The meaning of these equations is apparent. For the in-plane equations, the first and second equations (with respect to both derivatives) are compatibility and constitution together. The third and fourth (with respect to y_1) and the fourth and fifth (with respect to y_2) equations are equilibrium. For the out-of-plane equations, the sixth to eighth (with respect to both derivatives) in appropriate combinations are compatibility and constitution combined. The ninth and tenth equations referring to derivatives with respect to y_1 (or equivalently the tenth and eleventh equations referring to the derivatives with respect to y_2) are equilibrium. The remaining equations for both in-plane and out-of-plane ensure consistency with the original two second order and one fourth order equations.

These equations are now in the standard form of type II. The general process of obtaining (c.3.8), (c.3.7) and (c.3.7)' will now be outlined. (The process is a generalisation to distributed parameter system models type II of the lumped parameter example of §A.2.)

(i) For the original system equations of total order cn in the behaviour variables (with respect to the independent variables y_k, $k = 1, \ldots, c$; $c \leq 3$), irrespective of the order in the geometry variables, cn state variables are

introduced.
For the shell equations (c.3.4), w_1 and w_2 are to the second order and w_3 to the fourth order with respect to both y_1 and y_2. That is, sixteen state variables result.

(ii) The cn state variables are chosen related to increasing derivatives with respect to all the independent variables. The process of obtaining these state variables is similar to that outlined for type I with the extension here to derivatives over all the independent variables and not just one independent variable (as in type I).
For example consider $(c.3.4)^3$, the state vector has one component related to the 0'th order of w_3 (namely x_6), two components related to the first order (x_7 and x_8), three components to the 2nd order (x_9, x_{10} and x_{11}), and so on. Notice in this last mentioned case the mixed y_1y_2 derivative was introduced to complete the three second order derivatives. When the same process is repeated on $(c.3.4)^1$, there results a state (x_1) to the 0'th order in w_1 and two states (x_3 and x_4) to the 1st order. For $(c.3.4)^2$, the state to the 0'th order in w_2 is x_2, and to the 1st order are x_4 and x_5. Notice that x_4 is common to the reductions of $(c.3.4)^1$ and $(c.3.4)^2$ and hence the total number of states for (c.3.4) was reduced from sixteen to fifteen.

(iii) The detail of the state variables is adjusted as for the type I reduction with the same qualifications but here extended to the directions of all independent variables.
For $(c.3.4)^3$, the states may be interpreted as deflection, slope, moments and shearing forces. (Two variables, x_{12} and x_{15} do not however fulfill the requirement of having accepted appellations although they may be given physical meaning. Their presence ensures that the state equations are equivalent to the original system equations.) Similar interpretations may be given to the in-plane portions of (c.3.4).

(iv) Auxiliary controls are introduced as the derivatives (with respect to all the independent variables) of the states with the highest order derivatives. For an equation of order cn_j in a behaviour variable (with respect to all y_k, $k = 1, \ldots, c$; $c \leq 3$), there will be n_j states of the highest order n_j-1. Thus cn_j-1 auxiliary controls are

introduced, being derivatives of these n_j states with respect to all y_k, $k = 1, \ldots, c$. The remaining derivative of the state (that is the difference between cn_j, the total number of possible derivatives of state, and $cn_j - 1$, the number of auxiliary controls) becomes the original system equation (but now using the newly introduced state and control notation). The choice of the state that receives this individual treatment is arbitrary, provided on differentiation it leads to the system equation. The presence of the auxiliary controls ensures equivalence of the original system equation and the state equations. They occupy the base (α=4) of the pyramid previously mentioned.
These ideas are perhaps easier to see in the illustration. Consider $(c.3.4)^3$ (total order in w_3 is eight with respect to y_1 and y_2). The highest order derivative states are $x_{12}, \ldots, x_{15}$ and are all third order in w_3. Therefore 8-1 auxiliary controls $u_5, \ldots, u_{11}$ were introduced and are all fourth order in w_3. The remaining derivative of state, namely $\frac{\partial x_4}{\partial y_2}$ was set equal to the system equation $(c.3.4)^3$ and became $(c.3.8b)^{14}$.

(v) The conventional controls choose themselves. They represent the physical properties.
For the example they were the extensional and bending stiffnesses.

In the illustration at hand, a reduction to equations type II avoided the choice (as was necessary for type I) of the independent variable required in the differentiation on the left hand side by having separate derivatives of both y_1 and y_2 on the left hand side. For these symmetrical equations, such a reduction may be the more favourable over type I. It is seen that the results are applicable for variable thickness shells. Also the twisting term in the out-of-plane contribution $(c.3.4)^3$ to equations (c.3.4) splits neatly between derivatives with respect to y_1 and y_2, and the Kirchhoff 'transverse force' becomes the more usual out-of-plane shearing force.

Generally any partial differential equation defined over a three dimensional spatial region may be reduced to the type II form, albeit with an increase in the number of dependent variables. This large increase in the number of variables (and the consequent rise in the number of equations that have to be handled, although admittedly of low order) appears to be the main objection to the type II form. Associated with this is a certain repetition of information (in the form of common equations) within the state equation representation. The use of auxiliary control variables also does not appeal but this time in a physical rather than a mathematical sense. Its advantages appear to lie in the treatment of structures whose behaviour is similar (though not necessarily the same) in the independent variable directions. For unsymmetrical equations and for economy in computations, the type I representation would be sought.

Consider the same illustration, but now interpreted in the standard form of type III. This equation type relies on the symmetry of the structure (yet in a different manner to type II) by emphasizing common cross derivatives. It is anticipated from inspection of the composition of the in-plane shearing term and the out-of-plane twisting term occurring in the shell equations (c.3.4) that the type III representation will be better suited to the out-of-plane portion compared with the in-plane portion. This statement will be amplified following the illustration. Consider the new states

$$x_1 \triangleq w_1 \qquad x_2 \triangleq w_2$$

(c.3.9)

$$x_3 \triangleq w_3 \qquad x_4 \triangleq K \frac{\partial^2 w_3}{\partial y_1 \partial y_2}$$

with controls $u_1 \triangleq D$, $u_2 \triangleq K$ as before. The states clearly represent the deformations in each of the coordinate directions and the twisting moment.

Taking the cross derivatives of the states $x_1, \ldots, x_4$ with respect to y_1 and y_2 results in the state equations;

(c.3.10)

$$\begin{bmatrix} \dfrac{\partial^2 x_1}{\partial y_1 \partial y_2} \\[2ex] \dfrac{\partial^2 x_2}{\partial y_1 \partial y_2} \\[2ex] \dfrac{\partial^2 x_3}{\partial y_1 \partial y_2} \end{bmatrix} = \begin{bmatrix} -\dfrac{2}{u_1}\left\{q_2 + \dfrac{\partial}{\partial y_2}\left[u_1\left(\dfrac{\partial x_2}{\partial y_2} - x_3 z_{22}\right)\right]\right\} - \dfrac{\partial}{\partial y_1}\left(\dfrac{\partial x_2}{\partial y_1} - 2x_3 z_{12}\right) \\[2ex] -\dfrac{2}{u_1}\left\{q_1 + \dfrac{\partial}{\partial y_1}\left[u_1\left(\dfrac{\partial x_1}{\partial y_1} - x_3 z_{11}\right)\right]\right\} - \dfrac{\partial}{\partial y_2}\left(\dfrac{\partial x_1}{\partial y_2} - 2x_3 z_{12}\right) \\[2ex] \dfrac{x_4}{u_2} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial^2 x_4}{\partial y_1 \partial y_2} \end{bmatrix} \quad \begin{bmatrix} \dfrac{1}{2}\left\{q_3 + \dfrac{\partial^2}{\partial y_1^2}\left(- u_2 \dfrac{\partial^2 x_3}{\partial y_1^2}\right) + \dfrac{\partial^2}{\partial y_2^2}\left(- u_2 \dfrac{\partial^2 x_3}{\partial y_2^2}\right)\right. \\ + z_{11}u_1\left(\dfrac{\partial x_1}{\partial y_1} - x_3 z_{11}\right) + z_{12}u_1\left(\dfrac{\partial x_1}{\partial y_2} + \dfrac{\partial x_2}{\partial y_1} - 2x_3 z_{12}\right) \\ \left. + z_{22}u_1\left(\dfrac{\partial x_2}{\partial y_2} - x_3 z_{22}\right)\right\} \end{bmatrix}$$

The main criticism of the form of (c.3.10) is that it does not contain enough information as it only highlights the cross derivative terms appearing in the original high order equations (c.3.4). It is remarked that to obtain the first and second state equations it had to be assumed that the control u_1 was constant (in order to obtain mixed state derivative terms free of control derivative terms which are to an odd order) whereas the third and fourth equations are for general u_1 and u_2 (as they contain mixed state derivative terms directly). The total set of equations is only of a type III form for constant controls. A general breakdown to a type III form is impossible where the highest order of any dependent variable is odd as occurred in the in-plane equations for D above.

Briefly, the outline to the above reduction is as follows. The similarities with the above reduction outlines for types I and II will be apparent and will not be emphasized.

(i) For a system equation of order 2p in its mixed derivatives of the behaviour variable, irrespective of the order in the geometry variables, p state variables are introduced.
For the example (c.3.4), p=1 in $(c.3.4)^1$ and $(c.3.4)^2$ and p=2 in $(c.3.4)^3$, leading to four state variables in total.

(ii) The states are chosen related to the cross derivatives of the behaviour variable with the first state variable of 0'th order and the last of order (2p-2). The state variables thus differ by order 2 in their derivatives such that when the mixed derivatives of state are taken, there results an ordered hierarchy of state equations.
For the example, considering $(c.3.4)^3$, the first state variable x_3 is 0'th order in w_3, the last state variable x_4 is 2nd order in w_3.

(iii) The detail of the states is modified to agree in form with a meaningful quantity, keeping in mind that the resulting state equations are required to be equivalent to the original system equations. For this equation type, as compared with I and II, it appears that no clearly defined constitution - compatibility - equilibrium breakdown is possible.
Considering the example, the states x_1, x_2, x_3 and x_4 may clearly be interpreted as deformations and twisting moment.

(iv) The controls choose themselves. They represent the physical properties of the structure.
In the example they are the extensional and bending stiffnesses.

C.3.1.8 General comments. Notice that by choosing the state variables and equations to satisfy physical motives in the above example, the type I representation has been restricted to applications involving constant or finite dimensional controls and the type III representation has been restricted to constant controls (principally in order that derivatives of the control do not appear on the right hand sides). In structures where such cases occur, and completely general controls are still required, it will be found necessary to choose state variables and state equations without total physical significance.

A reduction of the same shell equations to a state-control form of the types I and III satisfying the mathematics only, could be carried out according to the algorithms in appendix one. For type I, in addition to the unusual meaning of several of the variables and state equations, there would result eleven state equations (compared with eight above). Eleven is the sum of the orders of the y_1 derivatives in w_i, i = 1, 2, 3 (eight), D (one) and K (two). In general for a constitutive equation n'th order in a behaviour variable and r'th order in a geometry variable, there will result (n+r) first order state equations. For a reduction to a state-control form of the equation type II, satisfying the mathematics only (as for example in Armand 1972), there would result fifty state equations (compared with thirty above). Again the increase in the number of equations may be attributed to the terms which are derivatives of the geometry variable. For equations type III with all derivatives fully expanded, there would result an additional state variable (and hence an additional state equation) for the out-of-plane portion while the in-plane portion contains derivatives in one variable whose highest order is odd and clearly cannot be reduced to the even derivative form required by a type III format.

Between the two reduction schemes outlined (namely one motivated by giving meaning to the variables and equations concerned (illustrated with reference to the shallow shell equations above) - the other satisfying the mathematics alone without regard to meaning (the algorithms of appendix one)) there exist various schemes with combinations of qualities borrowed from these two schools of approach. These will not be outlined here, but instead various alternative reduction schemes with associated discussions may be found in the design example sections (§G,§I and §K) of part 2.

The two distinct reduction proposals would appear to be the fundamental reduction schemes leading to the lowest and highest number of state equations respectively. Conglomerate schemes borrowing ideas from both have an intermediate number. The validity of any reduction scheme can only be verified by showing the equivalence of the low order state equations with the original high order equation. The requirement of meaningful choices of state and control according to physical arguments can apparently be put to one side in formulating any reduction scheme.

One final note on the nonuniqueness of a set of state variables is required. It is apparent that within each equation type (I, II or III) a set of state variables may be associated with a given high order system model equation in many ways (for a given set of controls). The most desirable is the choice motivated by giving meaning to variables involved leading to equations of equilibrium, compatibility and constitution. However for any set of state variables $x_1, x_2, \ldots, x_n$ satisfying the definition of state, and for a given model and set of controls, it is possible to construct another set of state variables as functions of $x_1, x_2, \ldots, x_n$. Formally, a new set of state variables may be written

$$x_i^* = X_i(\underline{x}) \qquad i = 1, 2, \ldots, n$$

provided there exists a unique (nonsingular) transformation between the set of values $\underline{x}^*$ and $\underline{x}$. This is tantamount to changing the coordinate system in the state space (see for example Lanczos 1949, Synge 1960). As anticipated the resulting state equations are altered; this may or may not be desirable from a computational viewpoint.

The convenience of using the state variable form is apparent. Apart from the appropriateness physically of the state variables to the description of the internal composition of the model in certain cases, the resulting set of equations reduces to a form tractable to machine computation, while with the use of matrix and vector notation, the mathematics becomes very elegant indeed. The standard form ((c.3.1), (c.3.2) or (c.3.3)) is appealing with regard to the preparation of standard solving routines, while their first order property (second order for type III) is far more suitable to solution techniques than higher order equations. All equations are amenable to a reduction to the standard form of either (c.3.1), (c.3.2) or (c.3.3) (and in certain cases to all three) by a suitable choice of new (state) variables, facilitating a general discussion for all systems. Simultaneous equations are additive. The size of the state vector and state equations are increased accordingly (that is the sum of the state equations corresponding to each individual high order equation).

Recalling the lumped parameter decomposition procedure introduced in section §A, it was noted that state variables could be chosen as meaningful quantities while the resulting state equations were applicable for general (variable) controls. The extension of these traits to distributed parameter decomposition procedures was only possible for the type II models. For type I and III models, only one of these traits could be satisfied at a time. The inference to draw from this is that it may be preferable to discretize distributed parameter models, before starting computations, in all but one independent variable direction such that the designer is then working with a lumped parameter form. (See section §E.) (With discretization comes the additional gain of simplified computations. In particular, lumped parameter system designs are an order of magnitude less difficult than distributed parameter system design.) For system model manipulations (such as much analysis, estimation and other procedures) and for the modelling of structures with constant controls, the distributed parameter form could be used directly while preserving physical meaning of all variables and equations.

C.3.1.9 Probabilistic equivalent: The state at any y (spatial or temporal coordinate) in stochastic models is regarded as the information that uniquely determines the probability distribution of the state at any other y = y' (§A.2). It is evident therefore that the state, as defined for example in the lumped parameter case,

$$\frac{d\underline{x}(y)}{dy} = \underline{f}(\underline{x}(y),\ \underline{u}(y),\ y) \qquad \text{(a.2.4)}$$

represents a stochastic process 'without after effect', that is a Markov process. This Markovian property is directly attributable to the way in which the concept of state has been defined. Markov processes are the probabilistic equivalent of the deterministic principles of (classical) mechanics - this property is sometimes referred to as the generalised causality principle; for evolutionary processes, the future may be predicted from a knowledge of the present alone.

By definition, a stochastic process $\{\underline{x}(y);\ y \in Y\}$ is a (real, vector-valued) Markov process if for every finite set of values $y^0 < y^1 < \ldots < y^{k-1} < y^k$ in Y

$$P\{\underline{x}(y^k) < \underline{\xi}^k | \underline{x}(y^i) = \underline{\xi}^i,\ i = 0,\ 1,\ \ldots\ ,\ k-1\}$$

$$= P\{\underline{x}(y^k) < \underline{\xi}^k | \underline{x}(y^{k-1}) = \underline{\xi}^{k-1}\}$$

The term on the right hand side of this equation is referred to as the 'transition probability distribution function' or simply the 'transition function'. It does not depend on values of $\underline{x}$ previous to y^{k-1}. A markov process is completely defined by specifying the absolute probability distribution $F(\underline{x}^0)$ and the transition probability distributions (see for example Doob 1953, Bharucha-Reid 1960).

Note that the stochastic processes representing the state may not in general be Markov. The probabilistic form of the state has been defined in this manner in analogy with the deterministic definition of the state. Markov processes imply a form of dependency between states at successive y values but not total dependency as may be anticipated in structural applications. The Markov assumption has been introduced in order to obtain solutions. It represents a compromise with the complete stochastic treatment where a solution is almost certainly unattainable. (The difficulties of the computations in existing structural thinking in even simplified stochastic cases are enormous - see for example Bolotin 1966, 1972, Vorovich 1966.) The theory of Markov processes is quite well delineated and this theory may be freely used to simplify stochastic calculations.

C.3.2 End-state conditions. To completely define the state throughout the model, certain 'starting' values of the state are required. These values are conventionally termed boundary and terminal conditions (here collectively referred to as end-state conditions) and may be expressed as conditions of state specified at the left and/or right interval limits (y_i^L and y_i^R respectively) of the independent variables $\{y_i;\ i = 1, \ldots\}$. The state at y_i^L and y_i^R will be required to belong to a given set of states S^L and S^R respectively. (The term 'end' is used in a 'left' and 'right' sense and not in an evolutionary sense, while the terms 'left' and 'right' imply an orientation of the coordinate space relative to the reader.)

Structural problems will generally be described with end-state sets S^L and S^R prescribed respectively by

(i) At y_i^L, $i = 1, \ldots, 4$; the intersection of p surfaces whose equations are

$$\text{(c.3.11a)} \qquad S_\alpha^L(\underline{x}, \ldots, \partial_{\underline{\ell}}\underline{x}, \ldots) = 0 \qquad \begin{array}{l} \alpha = 1, 2, \ldots, p \\ 0 \leq p \leq n \end{array}$$

(ii) At y_i^R, $i = 1, \ldots, 4$; the intersection of q surfaces whose equations are

$$\text{(c.3.11b)} \qquad S_\beta^R(\underline{x}, \ldots, \partial_{\underline{\ell}}\underline{x}, \ldots) = 0 \qquad \begin{array}{l} \beta = 1, 2, \ldots, q \\ 0 \leq q \leq n \end{array}$$

where for a well defined problem, $p+q = n$ (that is, the total number of end-state conditions equals the number of state equations). In (c.3.11), $\underline{\ell}$ is determined by the state equation type (that is I, II or III) (for example in I, $\underline{\ell} = (\ell_j, \ell_k, \ell_h)$; $j, k, h(\neq i) = 1, \ldots, 4$) and $\partial_{\underline{\ell}}\underline{x}$ is as defined in the 'notation'.

As an illustration, consider the boundary conditions along the free edge $y_1 = a$ of a shell (Timoshenko and Woinowski-Krieger 1959, Flugge 1973). For zero Poisson's ratio they read

$$- \frac{\partial}{\partial y_1}\left(K \frac{\partial^2 w_3(y_1,y_2)}{\partial y_1^2}\right) - 2\frac{\partial}{\partial y_2}\left(K \frac{\partial^2 w_3(y_1,y_2)}{\partial y_1 \partial y_2}\right)\Bigg|_{y_1 = a} = 0$$

$$- K \frac{\partial^2 w_3(y_1,y_2)}{\partial y_1^2}\Bigg|_{y_1 = a} = 0$$

(c.3.12)

$$D\left[\frac{\partial w_1(y_1,y_2)}{\partial y_1} - w_3 z_{11}\right]_{y_1 = a} = 0$$

$$\frac{D}{2}\left[\frac{\partial w_1(y_1,y_2)}{\partial y_2} + \frac{\partial w_2(y_1,y_2)}{\partial y_1} - 2w_3 z_{12}\right]_{y_1 = a} = 0$$

representing conditions of zero transverse shearing force (the Kirchhoff condition), zero bending moment, zero normal in-plane force and zero in-plane shearing force.

For state equations type I, setting the controls $u_1 \hat{=} D$, $u_2 \hat{=} K$ and the states as in equations (c.3.5), then these end conditions become

$$-x_8\Big|_{y_1 = a} = 0, \qquad -x_7\Big|_{y_1 = a} = 0$$

(c.3.13a)

$$x_2\Big|_{y_1 = a} = 0, \qquad x_4\Big|_{y_1 = a} = 0$$

respectively. For state equations type II, with the same controls but now with the states as in equations (c.3.7), the same end conditions become

$$-x_8 - \frac{\partial x_{10}}{\partial y_2}\Bigg|_{y_1 = a} = 0, \qquad -x_9\Big|_{y_1 = a} = 0$$

(c.3.13b) $$x_3\Big|_{y_1 = a} = 0 \qquad x_4\Big|_{y_1 = a} = 0$$

Similarly (but here for constant u_1, u_2) for state equations type III using (c.3.9), then

(c.3.13c)
$$-\frac{\partial^3 x_3}{\partial y_1^3} - 2\frac{\partial x_4}{\partial y_2}\Bigg|_{y_1 = a} = 0, \qquad -\frac{\partial^2 x_3}{\partial y_1^2}\Bigg|_{y_1 = a} = 0$$

$$\frac{\partial x_1}{\partial y_1} - x_3 z_{11}\Bigg|_{y_1 = a} = 0 \qquad \frac{\partial x_1}{\partial y_2} + \frac{\partial x_2}{\partial y_1} - 2x_3 z_{12}\Bigg|_{y_1 = a} = 0$$

In dynamic problems, over the time interval, S^L and S^R represent initial and final (or collectively, terminal) conditions on the state. Note that the inconsistency of specifying the final state without ensuring the controllability of the model or whether that state is attainable, should be guarded against.

The above formulation for (c.3.11a,b) includes the case of end-state constraints. The question of state constraints is dealt with in the following section (§D).

Random end-state conditions. For random state variables the end-state conditions will be given in terms of certain probabilistic characteristics of these variables - for example probability densities: $p[\underline{x}(y^L)]$ is the probability density function of the state at y^L such that $p[\underline{x}(y^L)]\,d\underline{x}(y^L)$ is the probability that the state $\underline{x}(y^L)$ is contained within the elemental volume $d\underline{x}(y^L)$ (= $dx_1(y^L) \ldots dx_n(y^L)$) about $\underline{x}(y^L)$.

Deterministic end-state conditions may be considered as special cases of the random specification. For example the probability density function becomes the Dirac delta function; for $\underline{x}(y^L) = \underline{c}$, a fixed vector of constants, then $p[x_i(y^L)] = \delta(x_i(y^L) - c_i)$, $i = 1, \ldots, n$.

C.3.3 Response transformation. The response will be related to the state through an algebraic equation of the form (c.3.14) $\underline{z}(\underline{y}) = \underline{h}[\underline{x}(\underline{y}), \underline{y}]$

(see for example Porter 1969) where $\underline{h} = (h_1, \ldots, h_m)^T$ in general is a nonlinear vector-valued function of the arguments shown. Equation (c.3.14) includes the case where the state and response bear a one-to-one relationship.

C.3.4 Comment. It will be apparent that the state equations given by (c.3.1) (or c.3.2 or c.3.3) and the end-state conditions (c.3.11) (together with an appropriate response transformation), represent a set of equations where the number of unknowns (n+r) exceeds the number of equations (n) as the r controls are still unspecified.

For given controls the equations may be solved for the unknown states. This is the familiar analysis problem. By comparison the synthesis problem makes no a priori assumptions as to the form of the control. In general for a specified performance (state) there will not exist a unique solution and a means is required of directly selecting a control such that the system model not only performs as specified but is optimal in some sense. A performance (or design) index (or optimality criterion) provides the basis for a unique and at the same time meaningful solution. A value or rating of the index corresponds with each feasible solution (control) and from which the optimum solution may be chosen.

As control theory established the relationships under which a system model may be controlled, optimal control theory establishes a particular control according to a given criterion. It is the subject of optimal control (in a structures sense) which is taken up in the next section.

A note on terminology: In the following presentation the terminology 'system model type I (or II or III)' will refer to a system model with a control-state description of type I (or II or III) together with appropriate end-state conditions and response transformation. The end-state conditions and response transformation will often be omitted in the discussion of the model but it is emphasized that their specification in a suitable form is implied. This will facilitate subsequent developments of the modelling procedures. It also will be found semantically convenient in the following sections to use the terminology 'system model', 'system' and 'model' synonymously although a distinction between the abstract mathematical model and the real physical system is always intended. The model is only ever a representation of the system.

§D OPTIMAL CONTROL AND THE COMPONENTS OF A DESIGN PROBLEM

D.1 OPTIMAL CONTROL.

The problem of developing an optimal system is one of finding an admissible control such that the system functions according to the design objective. A more formal statement will be given later. The design objective or criterion is expressed analytically as a functional with, in general, both state and control variable arguments. Certain physical, operational and engineering constraints may be present, restricting the control choice. This choice may also be simplified if the search is confined to certain classes of systems (for example shell action, plate action). The problem is thus one of choosing the system control such that the system operates in some best way, while observing the constraints present. An extremisation of the criterion functional is implied.

The formulation of an optimal control problem requires the following components.

(i) A model of the system to be controlled. (refer §C.3) This is the constitutive equation (together with end-state conditions and response transformation) ideally expressed in state equation form in preference to a single high order equation. It characterizes the system and enables the effect of alternative controls on the system to be predicted.

(ii) The constraints upon the design. (§D.2) Constraints limit the range of permissible solutions and fix many of the system properties.

(iii) The demands presented to the system in the form of a design goal (objective, criterion or index). (§D.3) The criterion is derived from a design value statement. To evaluate possible alternative solutions, a scalar index is introduced. The problem is to determine the control that gives the least or greatest value of this index.

Solution controls are said to be feasible if they satisfy the system model and are within the permissible bounds as defined by the constraints. Where a range of feasible solutions exists the problem is considered well posed. The design goal provides the criterion by which the optimal control is chosen from the set of feasible controls in order that the constraints are satisfied in the best manner. A particularly thorough treatment of optimal control is given by Fel'dbaum (1965).

Using a state and control foundation, superficially different design problems may be shown to share a common mathematical basis, leading to common solution techniques (parts 2,3).

D.2 DESIGN CONSTRAINTS.

D.2.1 Introduction. Constraints influence the design solution characteristics by isolating admissible solutions from all possible solutions, and give meaning to the choice of the optimal system. Constraints may be defined on certain subsets, boundaries or throughout the Y domain and will be given in the form of inequalities or equalities. The role of constraints in design has been outlined by Lee 1964, Bellman 1957, Tsypkin 1971, Fel'dbaum 1965, among others.

In a sense the previous system equations may be regarded as (differential) equality constraints over the total Y domain. The system is constrained to belong to the class of systems whose constitutive relationships are of this form. Terminal and boundary conditions may be likewise treated as equality constraints at $t^{L,R}$ and on ∂D respectively.

Constraints typically restrict the freedom of location of the state and control variables in their respective function spaces. The range of possible values that the states and controls may assume is reduced to a set of admissible values. (The idea of admissibility will be later extended to include allowable classes of functions.)

The following three subarticles categorize the constraints according to whether they relate to the control (§D.2.2), state (§D.2.3) or combined control and state (§D.2.4). A further subarticle (§D.2.5) on reliability constraints has been included because of the present popularity of the subject in the structures literature, although it is shown that it falls within the previous constraint categories.

D.2.2 Constraints on the control. Geometric, material and related physical properties, functional and aesthetic considerations of the design restrict the choice of the control vector $\underline{u}$ to lie within a set or region U in the space of the control.

$$\underline{u}(\underline{y}) \in U \qquad\qquad \forall\ \underline{y} \in Y$$

The admissible region U may vary with the parameters $\{y_i;\ i = 1, \ldots\}$

$$U = U(\underline{y})$$

Controls satisfying the above requirement are termed admissible controls. Of particular interest will be constraints such that

$$U \triangleq \{\underline{u}\ ;\ h_j^{(1)}[\underline{y},\ \underline{u}(\underline{y})] \leq 0,\ j = 1, 2, \ldots, m^{(1)}\} \tag{d.2.1}$$

$$m^{(1)} \leq r \text{ for equality constraint}$$

where $h_j^{(1)}$ are prescribed functions of the arguments shown. The sense of the inequality is taken as less than or equal to zero without loss of generality. (An extended definition of U having state arguments in addition to the parameters y_i has been considered by Berkovitz 1961 but its usage is uncertain in the structures case.)

As an illustration, physical limitations may restrict the rigidity, u_i, of structural members $\{i;\ i = 1, \ldots, r\}$, for all $\underline{y}$, in which case the constraint assumes the form

$$0 \leq u_i \leq a_i \qquad\qquad i = 1, \ldots, r$$

or

$$(u_i - a_i) \leq 0 \qquad , \qquad -u_i \leq 0$$

where the a_i are prescribed. The admissible region U in this case is a closed, bounded and convex set. In finite dimensions it may be geometrically interpreted as an r-dimensional polyhedron.

D.2.3 Constraints on the state. Similar bounds on the coordinates $x_1, \ldots x_n$ of the state vector may be expressed;

$$\underline{x}(\underline{y}) \in X \qquad\qquad \forall\ \underline{y} \in Y$$

where

$$X = X(\underline{y})$$

The state is constrained to lie within the set or region X defined in the state space. The admissible region will usually take the form

$$(d.2.2) \qquad X \mathrel{\hat=} \{\underline{x} \; ; \; h_k^{(2)}(\underline{y}, \underline{x}(\underline{y}), \ldots, \partial_{\underline{\ell}}\underline{x}(\underline{y}), \ldots) \leq 0, \; k = 1, 2, \ldots \}$$

where $\underline{\ell} = (\ell_1 \ldots)$ and $h_k^{(2)}$ are prescribed functions of the arguments shown.

State constraints represent imposed limitations on the system behaviour over space and time. The most obvious examples would be limitations on the structure displacements and rotations, internal moments and shearing forces, which include the notion of 'limit states' (C.E.B 1964 for example) as a special case.

In general the state will be required to satisfy certain end conditions (equality constraints) at the left and/or right interval limits of the independent variables $\{y_i ; i=1, \ldots\}$. These are commonly termed state boundary and terminal conditions (§C.3.2).

D.2.4 Mixed state and control constraints. Frequently constraints imposed on designs are not expressions of either $\underline{x}$ or $\underline{u}$ alone. Rather, functions of both the state and control may be restricted to some admissible region V say, defined on the state-control product space. Analogous to the previous constraint formalisms, of most concern will be admissible regions

$$(d.2.3) \qquad V \mathrel{\hat=} \{(\underline{x},\underline{u}) \; ; \; h_i^{(3)}[\underline{y}, \underline{x}(\underline{y}), \ldots, \partial_{\underline{\ell}}\underline{x}(\underline{y}), \ldots, \underline{u}(\underline{y})] \leq 0,$$

$$i = 1, 2, \ldots, m^{(3)}\}$$

$$m^{(3)} \leq r \text{ for equality constraints}$$

where the ordered pair $(\underline{x},\underline{u}) \in V$, $V = V(\underline{y})$, $h_i^{(3)}$ are functions of the given arguments, and $\underline{\ell} = (\ell_1, \ldots)$.

Existing models for optimum structural design classify their design constraints according to whether they are 'geometric' or 'behavioural'. (See for example Rozvany 1966, Sheu and Prager 1968.) The classification terms suggest that the equivalent constraints would generally be grouped under the present 'control' and 'state' constraints respectively. Usually no distinction is made in existing design models for constraints which are combinations of the two types of constraints (that is geometric-behavioural), possibly because the associated design techniques are not strongly influenced by classifications in a design model. The techniques to be outlined in parts 2 and 3 however reflect, and in fact are determined by, the form of any constraints present.

Where the system is stochastic, (d.2.1) to (d.2.3) may be replaced by expressions giving the mathematical expectations of the constraint. For example (d.2.3) becomes

$$M\{h_i^{(3)}[\underline{y}, \underline{x}(\underline{y}), \ldots, \partial_{\underline{\ell}}\underline{x}(\underline{y}), \ldots, \underline{u}(\underline{y})]\} \leq 0 \quad i = 1, 2, \ldots, m^{(3)}$$

D.2.5 Reliability constraint. The requirement that a structure function without 'failure' is fundamental. 'Failure' is implied in the sense of exceeding a certain limit state, corresponding, for example, to measures of unserviceability or instability. (See for example C.E.B. 1964, Rowe 1970.)

For stochastic systems, a scalar-valued constraint, the system reliability, is employed to ensure the successful functioning of the system. (See for example Julian 1957, Freudenthal et al 1966, Borges and Castanheta 1968, 1972, Pugsley 1966.) The equivalent constraint for deterministic systems is embodied in the concept of 'factor of safety'.

A detailed treatment of reliability analysis using the present modelling procedures is outlined in appendix two. Reliability constraints are shown to be mixed state-control constraints or state-only constraints. Reliability as a design criterion is considered in the following article.

D.3 DESIGN OPTIMALITY CRITERIA.

Optimality criteria provide the means of quantitatively assessing alternative designs. The design solutions are only optimal in the sense of the criteria which follow from the design problem statements, although computational tractability reasons may warrant introducing alternative, simpler criteria. The latter criteria obviously lead to suboptimal designs with respect to the original criteria.

Optimality implies an extremization requirement on some measure Q, the optimality criterion. In general this measure will be a functional of both state and control functions and will be a scalar quantity

$$\hat{Q} \triangleq \text{ext} \; Q(\underline{x}, \underline{u}, \underline{y}) \qquad \forall \; \underline{y} \in Y \tag{d.3.1}$$

The criterion Q may be thought of as assigning a unique real number to each admissible solution. The optimum $\hat{Q}$ is chosen from the many feasible values of Q. Alternatively Q may be considered as a function in which the controls play the role of the independent variables. The criterion derives from an imposed value system, the correct identification of which remains essential for a meaningful design. Only its mathematical formulation is included here. Quantitative measures must replace qualitative (in subjective value systems) for the mathematical problem to exist.

Suboptimal control assumes an additional role to that mentioned in the opening paragraph of this article. In particular the implementation of the optimal control may be infeasible for engineering, economic or other reasons (that is other constraints not allowed for in the mathematics of the design problem). Knowing the optimal control enables the implementation of a suboptimal form with a full understanding of the consequences of such action. In this sense the optimal control serves as a design standard by which alternative controls may be evaluated.

Without loss of generality, minimisation will be implied in all optimization studies. It will be appreciated that any problem in maximisation may be conveniently treated as a problem in minimisation by means of a suitable negative transformation: max(-Q) = - min(Q).

Two analytically tractable parts of a criterion, covering a broad class of problems may be recognised. (These are essentially generalisations of the same quantities extremized in the classical calculus of variations. In particular note the correspondence with the Bolza problem.) Various design indices may be obtained by specializing, or applying suitable mathematical transformations to either. Non-analytic criteria are not considered initially. The two distinguishable parts are:

(i) A generalised domain criterion in which Q is a scalar quantity obtained by integrating over the domain of the independent variables. In notation consistent with the previous system modelling,

$$Q = \int_Y G[\underline{y},\ \underline{x}(\underline{y}),\ \ldots\ ,\ \partial_{\underline{\ell}}\underline{x}(\underline{y}),\ \ldots\ ,\ \underline{u}(\underline{y})]\,dY$$

where $\underline{\ell} = (\ell_1,\ \ldots\ )$. The integrand G is a prescribed scalar function of the arguments shown. $Y \subset E^{\alpha}$, $\alpha = 1, \ldots$ with coordinate vector $\underline{y} = (y_1,\ \ldots\ )^T$.

(ii) An end criterion expressing a general function of the states at the right and/or left interval limits of the independent variables $\{y_i;\ i = 1, \ldots\}$. For example, a so-called 'final criterion' is a function of the states at the right limit of the time interval $[t^L, t^R]$. Reverting to the original distinction between the time domain (T) and the space domain (D), then

$$Q = \int_D g[\underline{x}(\underline{y},t)]\Big|_{t = t^R} dD$$

where g is a scalar function of the states shown. Also, for example, a criterion defined on a closed boundary curve ∂D,

$$Q = \oint_{\partial D} g'[\underline{x}(\sigma)]\,d\sigma$$

where σ is a measure of arc length.

The domain criterion receives sufficient usage in structural applications to warrant its own treatment, although it can be transformed into an end criterion. This is illustrated in the following example. It is remarked that a very general criterion contains both domain and end criterion parts and hence, for this case, a separate treatment of each part is required.

As an example the domain criterion presented in (i) may be transformed into an end criterion by suitably augmenting the state space. In, for example, a type I format, by introducing an additional state coordinate $x_0(\underline{y})$, where

$$\frac{dx_0(\underline{y})}{dy_4} = f_0 = \int_D G[\underline{y},\ \underline{x}(\underline{y}),\ \ldots\ ,\ \partial_{\underline{\ell}}\underline{x}(\underline{y}),\ \ldots\ ,\ \underline{u}(\underline{y})]\,dD$$

with $x_0(y_1,\ y_2,\ y_3,\ y_4^L) = 0$, then minimising the domain criterion over the domain $Y = T \times D$ is equivalent to minimising $x_0(y_1,\ y_2,\ y_3,\ y_4^R)$. It is seen that this is a special case of (ii). Hence derivations based on the end criterion in (ii) will apply to the domain criterion in (i) combined with an augmented state space. This result will be used in the derivation of certain design optimality conditions in part 2.

For a discussion on transformations between various criteria see for example Tou (1964) among others. In most transformations, the dimensionality of the state vector is enlarged.

Probabilistic systems. For random values of its arguments, the general criterion Q of (d.3.1) is now a random quantity, and hence an unsuitable measure. A suitable deterministic measure, over which the minimisation may be carried out, is the expected value or first moment (in a probabilistic sense) of Q, $M\{Q\}$, where $M\{\cdot\}$ denotes the expectation operation. The expectation operation may be visualized as taking the average of the criterion evaluated for each of the possible values of its arguments.

In general, this expected value of the random measure Q, is used as the system criterion. However in certain applications a measure or index of reliability may be relevant; that is extremising the index may relate to minimising the probability of the structure exceeding (both positive and negative senses implied together or singly) a particular limit state, or maximising the probability of non-exceedance in order that the system attains a maximum level of reliability. This non-analytic case is treated at length by Gnedenko et al (1969) and Tsypkin (1971). Notice that this is a different situation to the one in which a system is designed for a given reliability (the probability of the state exceeding a given limit state is prescribed). Reliability in this context is a constraint (§D.2).

Comment: Several criteria (resulting from multiple requirements on a design) expressed for the one problem in general lead to different points in solution (control variable) space. In general the points do not coincide and hence the existence of more than one criterion simultaneously is inadmissible for a meaningful problem. Auxiliary conditions, equivalent to constraints, may however coexist with the optimality criterion. Adaptations of Lagrangian multiplier and weighting function concepts are invariably employed for a solution. (See for example Fel'dbaum 1965; also Zadeh 1958 for nonscalar-valued criteria and Kalman 1964 on the question of optimality.) Alternatively trade-offs or adjustments may be made between the several design requirements.

§E A STATEMENT OF THE DESIGN PROBLEM AND A VIEW TO ITS SOLUTION

E.1 STATEMENT OF THE OPTIMUM PROBLEM.

E.1.1 Outline. Using a state-control basis for the modelling of the components of the design problem, superficially different problems can be shown to share a common mathematical association. (The common conceptual basis has always existed.) A statement of the design problem and its solution may then be treated in unified modes.

The difficulty has existed in elaborating the last two sections, of achieving a formulation which is of sufficiently broad generality yet not so general as to prohibit the development of effective solution procedures. For the level of generality of the modelling of the previous two sections, the design problem may be stated as follows.

Deterministic. The simplest deterministic problem may be formulated as: To find the control $\underline{\hat{u}}(\underline{y})$ which minimizes the criterion functional Q(parts (i) and (ii)), for a system behaving according to equation (c.3.1) (or(c.3.2) or (c.3.3)) over a given domain Y, with end-state conditions (c.3.11). $\hat{u}$ is termed the optimal control.

Stochastic. The equivalent stochastic problem chooses the control to minimize the expected value of Q subject to a system equation, which is now a stochastic differential equation, and end-state conditions, which now have probabilistic characteristics. The extension, in principle, from the deterministic case is quite straightforward.

Provided the probability characteristics of the variables are known and given the basic Markovian assumption, it is apparent that there are no prominent dissimilarities in the deterministic and stochastic formulations. The solutions should also be similar. Stochastic solutions will only permit optimality on the average whereas deterministic solutions will be optimal for each case; this difference however evolves from the idea of probability rather than different formulations.

The problem outlined in the deterministic case is one of functional minimization while the problem class is of a generalized Bolza type (for multiple integrals and with side constraints) encountered in the calculus of variations. (See Tou 1964 for discussion.) Where part (ii) is identically zero in the Bolza problem, it is the problem of Lagrange. Where part (i) is identically zero in the Bolza problem, it is the problem of

Mayer. (See Bliss 1946, Bolza 1931, 1961 among others.) The various problem forms are interchangeable by suitably defining new variables.

Complications are added by the prescription of constraints on the admissible state and control regions - that is equations (d.2.1) to (d.2.3). For lumped parameter systems, the extension or generalisation of the calculus of variations to account for bounded controls is due to Pontryagin (Pontryagin et al 1962; see also Lee and Markus 1967, Leitmann 1966, Rozonoer 1959). More formally the extension is known as Pontryagin's maximum principle. (An equivalent minimum principle has appeared in some recent American texts. See for example Bryson and Ho 1969, Sage 1968 and the discussion in §J.) Only a formulation consistent with the original maximum principle of Russian literature will be considered in this thesis.

Extensions of the principle to general nonlinear distributed parameter systems are given in part 2 for system model types I, II and III, where the control-state composition of the system problem is used to advantage in the derivations. The necessary conditions for optimality in general constitute a set of partial differential equations nonlinear in both the state and control variables and of the boundary value type (end conditions are split), the solution of which may present certain complications. Mathematical tools available to handle nonlinear partial differential equations are at the present stage most inadequate.

The necessary optimality conditions for the problem with constraints on the state have also been obtained by Pontryagin et al (1962) for the lumped parameter case. Related work may be found in Berkovitz (1961, 1962) and Chang (1962) among others.

An alternative approach to the control optimization problem and one which extends readily to the stochastic case is Bellman's dynamic programming (Bellman 1957a, 1961, Bellman and Dreyfus 1962) based on the principle of optimality. In part 3, stochastic optimality conditions are derived in a systematic manner for the lumped parameter case and are shown to be equivalent to the results obtained by applying the principle of optimality to the problem. The conditions assume the form of a recurrence relation in $\mathop{Q}_{\sim}$ with the optimal control resulting as a by-product of the solution process. There are however, certain computation limitations which restrict the scope of application of the conditions.

These comments are amplified in the relevant sections in parts 2 and 3.

E.1.2 A note on the solution techniques used in part 2. Three distinct approaches to the derivation of distributed parameter versions of Pontryagin's maximum principle are given in part 2 for the three system model types. It is emphasized that the approaches are not exclusive to any one system type but are essentially complementary in that they all lead to similar necessary conditions. The assumptions involved in obtaining the conditions however vary.

Classical calculus of variations arguments are used for system type III in section §J. Here it is required to have free, global variations with additional assumptions on the smoothness and continuity of derivatives. Modifications are possible to remove these restrictions. The necessary conditions for type II systems (§H) are derived via the dynamic programming technique of Bellman. The basic assumption relates to smoothness and continuity conditions on the function expressing the minimization of the criterion. Using the variational arguments of Rozonoer (local variations), the above restrictions are not present. Section §F uses Rozonoer's approach to derive the necessary conditions for system type I. The backgrounds and bases for the approaches are discussed in the relevant sections (§J, §H and §F respectively).

In addition to the three approaches detailed above, two other routes to obtaining necessary conditions have been noted in the literature (in particular see Robinson 1971) and may be broadly classified as 'function space' and 'moment' methods. Moment methods are generally only applicable to linear integral equations with known eigenfunctions. (See Butkovskii 1969.) The more abstract function space methods illustrate a recent trend in the ways of obtaining necessary conditions. (See for example Butkovskii 1969, Wouk 1969, Neustadt 1969, Lions 1968.) However as commented by A.I. Egorov (1966), the introduction of abstract spaces imposes auxiliary constraints on the class of admissible controls not called for by the nature of the problem. Restrictions on the form of design criteria are also present.

It is noted that the earliest works in control on lumped parameter systems were variational in character, essentially being extensions of the classical calculus of variations. The original work in distributed parameter systems, as may have been anticipated, also employed variational arguments. Notable is the pioneering work of Butkovskii and Lerner (1960, 1961) and Butkovskii (1961, 1962, 1969). However they considered only systems modelled by integral equations and, unless a transformation is known between differential equations and integral equations, the work is inapplicable to the present system cases. For general nonlinear systems, a transformation is usually not available. Later contributions on systems described by integral equations (primarily linear) are by Khatri and Goodson (1966), Sakawa (1964, 1966), Yavin and Sivan (1967, 1968), Brogan (1968b) and Wang (1964) among others. However the integral equation form will not be developed further here for the reason given above. Systems described by integro-differential equations (for example Wittler and Shen 1969) will also not be discussed. Only differential systems will be considered further. For related background reading, reference should be made to the very complete surveys of Robinson (1971) and Butkovskii et al (1968) while Wang (1968) gives an extensive bibliography. Also complementary background articles may be found in sections §F, §H and §J.

E.2 SYSTEM APPROXIMATION.

To produce an analytically less complex system model, for purposes of solution of the design problem, various quantizing procedures may be employed. Essentially these involve either total or part discretization of the independent parameters $\{y_i ; i = 1, \ldots \}$ over their interval ranges $[y_i^L, y_i^R]$. (Discretization of the dependent functions - that is discretization in level - will not be considered. Variables may be classified according to whether they are discrete or continuous in their parameter set of level (that is, variable space). For example $x(y)$ may be discrete or continuous in (a) the parameter y or (b) the space Ξ, $x \in \Xi$. Equivalent distinctions exist for the stochastic case.)

Consider:

(i) Reducing the system equations to a form continuous in only one independent variable, (and hence discretely lumped in the other independent variables), offers the opportunity to exploit the use of the conceptually simpler lumped parameter solution techniques. All previous definitions, concepts and solution techniques have lumped parameter equivalents. The lumped parameter case is a commonly treated case in the literature owing to its conceptual simplicity compared with the distributed parameter case. (See Athans 1966, Robinson 1971 for discussion; Paiewonsky 1965 is also of interest.)

(ii) Total Y domain discretization yields a finite dimensional system of difference equations. The approximate system is then equivalent to a multistage system for which multistage decision processes may be used for the solution of the design problem. Most favoured among these are the discrete forms of the maximum principle (for example Chang 1960, Katz 1962, Halkin et al 1966) and dynamic programming (for example Bellman 1957a, 1961). The problem may also be formulated as one in nonlinear programming (for example Tabak and Kuo 1971) with the criterion Q assuming the form of a hypersurface in a suitable (usually control) space bounded by given constraints. Much of the control literature deals with this total Y domain discretization model as a result of its particular relevance to digital computation. Its use is particularly well documented. (See Fel'dbaum 1965 for discussion.)

A discretization scheme and notation for the discrete case to be adopted where the discrete form is required in the following sections may be summarized here: For the interval $[y^L, y^R]$ (the extension to a many dimensional Y domain will be apparent) subdivided into N equal intervals Δ, at any location $y = k\Delta$, $k = 0, 1, \dots, N$

$$\underline{x}(y) \rightarrow \underline{x}(k\Delta) \rightarrow \underline{x}^k$$

$$\underline{u}(y) \rightarrow \underline{u}(k\Delta) \rightarrow \underline{u}^k$$

The discrete values of the state, $\underline{x}^k$, apply at each discretization point k; the control $\underline{u}^k$ is held constant during each interval $[k\Delta, (k+1)\Delta]$. Derivatives may be replaced by their finite difference equivalents: At k,

$$\frac{d\underline{x}(y)}{dy} \rightarrow \frac{\underline{x}^{k+1} - \underline{x}^k}{\Delta} \; ; \; \frac{d^2\underline{x}(y)}{dy^2} \rightarrow \frac{\underline{x}^{k+1} - 2\underline{x}^k + \underline{x}^{k-1}}{\Delta^2} \; ; \; \dots$$

By so doing, differential equations reduce to difference equations and integrals to finite summations. (See for example Tabak and Kuo 1971, Daniel 1971.)

The behaviour of the discrete system and the original continuous system are assumed to be similar as the interval Δ goes to zero. The validity or consistency of the approximations, as well as questions of stability and convergence, may be found in Wang and Tung 1964 and Wang 1964 in particular.

As an aside, the sets of difference equations obtained through discretization bear a form which is reminiscent of that encountered in the transfer matrix technique of conventional structural analysis. (See for example Pestel and Leckie 1963, Livesley 1964.) To illustrate this, consider the state equations (a.2.3) for the beam manipulated in §A.2. The state derivatives may be replaced by their finite difference equivalents at k for equations (a.2.3) to read

$$x_1^{k+1} = x_1^k + \Delta x_2^k$$

$$x_2^{k+1} = x_2^k + \Delta \frac{x_3^k}{u^k}$$

$$x_3^{k+1} = x_3^k + \Delta x_4^k$$

$$x_4^{k+1} = x_4^k + \Delta q^k$$

or expressed in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & \Delta & & \\ & 1 & \Delta/u & \\ & & 1 & \Delta \\ & & & 1 \end{bmatrix}^k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^k + \begin{bmatrix} \\ \\ \\ \Delta q \end{bmatrix}^k$$

This matrix is clearly the transfer matrix for the particular interval [k, k+1] of the beam; for a beam of constant rigidity u = EI and length L, the entries in this matrix become 1, L, L/EI and 0. Transformations, for example as outlined in Livesley 1964, give the equivalent stiffness or flexibility matrices. The ease with which the above transfer matrix was derived could be compared with the rather involved existing methods as for example outlined in Pestel and Leckie among others. The extension to distributed parameter systems is apparent where the discrepancy between the above straightforward approach and existing approaches is even more marked.

<u>Discrete stochastic systems.</u> Discretizing the parameter set $Y \subset E^1$ such that $\{y^i;\ i = 0, 1, \dots, N\} \in Y$, allows a 'finite dimensional distribution' representation of the stochastic process. That is, the stochastic process $\{\underline{x}(y, \omega);\ y \in Y,\ \omega \in \Omega\}$ may be characterized by the joint distribution or joint density function

$$F(\underline{x}^0, \dots, \underline{x}^N) \text{ or } p(\underline{x}^0, \dots, \underline{x}^N) \quad \forall \{y^i\} \in Y$$

respectively (Kolmogorov 1931), where $\underline{x}$ is the conventional state column vector $(x_1, \dots, x_n)^T$. Equivalent descriptions apply for other vectors. The probability space will always be considered continuous and hence the density function will always exist. (This is not to be taken as a restriction on the approach, which is equally capable of handling discrete distributions, but rather delineates the scope of following sections. The extension to the treatment of discrete probability distributions will be apparent.)

Processes such as these, that is with a continuous probability space and discrete parameter set, may be referred to as 'random sequences' (the corresponding realization being referred to as a 'sample sequence'). The terminology 'stochastic process' is commonly saved for the case of continuous probability space, continuous parameter set (the corresponding realization being a 'sample function').

State equations with this discrete character are naturally referred to as stochastic difference equations. Stochastic difference equations may be regarded in a like manner to their deterministic equivalents; for a given control, the solution of either may be regarded as an algorithm defining (the joint probability distribution of) $\underline{x}^i$ recursively from the previous value $\underline{x}^{i-1}$. Under the assumptions made above for the concept of state, these processes $\{\underline{x}^i;\ i = 0, 1, \ldots, N\}$ are Markov processes with a finite number of states. (See Doob 1953, Wong 1971 and others.)

For a Markov process, the joint density function may be written

$$p(\underline{x}^0, \ldots, \underline{x}^N) = p(\underline{x}^N|\underline{x}^0, \ldots, \underline{x}^{N-1})\ p(\underline{x}^0, \ldots, \underline{x}^{N-1})$$

$$= p(\underline{x}^N|\underline{x}^{N-1})\ p(\underline{x}^0, \ldots, \underline{x}^{N-1})$$

Repeating the procedure, the density function reduces to

$$p(\underline{x}^0, \ldots, \underline{x}^N) = p(\underline{x}^0) \prod_{i=1}^{N} p(\underline{x}^i|\underline{x}^{i-1})$$

That is the probability law for a Markov process is completely determined by the two-dimensional distributions. The conditional function $p(\underline{x}^i|\underline{x}^{i-1})$ is referred to as the 'transition probability density' and denotes the probability that the system which had state $\underline{x}^{i-1}$ at y^{i-1} will for $y^i > y^{i-1}$ have a state $\underline{x}^i$. An equivalent expression relates the (N+1) dimensional probability distribution function $F(\underline{x}^0, \ldots, \underline{x}^N)$ to two-dimensional distributions.

PART 2

DETERMINISTIC DESIGN

§F DERIVATION OF NECESSARY CONDITIONS FOR OPTIMALITY: SYSTEM TYPE I

F.1 INTRODUCTION

F.1.1 Outline. Part 2, comprising sections §F to §L, develops the techniques for dealing with the design problem within the conceptual framework offered by control systems theory. For the three system models considered (namely types I, II and III), distributed parameter versions of Pontryagin's maximum principle are derived (sections §F, §H and §J respectively). The approach to optimum design, based on the maximum principle, does not give an explicit expression for the optimal design but instead optimality is manifested in a set of necessary conditions that have to be satisfied. The basic results are demonstrated (sections §G, §I and §K respectively) on a common illustration which also serves to highlight the basic modelling techniques of part 1. The section groupings §F-§G, §H-§I, and §J-§K, being associated with the system model types I, II and III respectively, may be considered independently of each other. (The sequencing of the sections is not intended to imply an order of reading.) Singular formulations of design problems are considered in the last section (§L) of part 2 and the treatment is applicable to all model types.

The most powerful technique for the solution of deterministic optimal control problems as outlined in section §E, is the maximum principle of Pontryagin (Pontryagin et al 1962; see also the proofs of Rozonoer 1959, Halkin 1963, Lee and Markus 1967, among others). In essence the principle extends the results of the calculus of variations to include constraints on the control in the problem statement (Berkovitz 1961). The presence of these constraints prevents the direct use of the results of the classical calculus of variations. (Nevertheless without these constraints, the calculus of variations would be a less convenient though adequate solution technique. Extensions to the basic calculus of variations theory are also available to handle constraints, discontinuities and the like.) For general lumped parameter systems, the maximum principle can be shown to be a set of necessary conditions, while in certain restricted cases (for example linear systems) sufficiency can be shown also (Pontryagin et al 1962).

An extension of the classic results of Pontryagin is detailed in this section for distributed parameter systems of type I. (See §H for systems type II, §J for type III.) The derivation of the extension of Pontryagin's results follows a course of arguments analogous to the independent derivation of the maximum principle for lumped parameter systems given by Rozonoer (1959). The derivation is of a variational calculus style (local variations). Piecewise continuous controls are allowed, there being no smoothness assumptions as required for the global variations used in the classical calculus of variations. The solution is effected by considering a variation of a functional and obtaining estimates for the terms involved. The lumped parameter system results of Rozonoer may be identified as a special case.

For the derivation, it will be found convenient in the initial studies to consider a certain fundamental problem (§F.2) and then to extend or adapt the derivation (§F.3) so as to eventually cover a broad class of optimization problems. In such an approach there lies the difficulty of formulating a problem with sufficient generality to cover a broad class of problems yet narrow enough to allow an effective solution technique. This form of attack on the design problem maintains a tractable level of computations throughout. It is shown (§F.3) that the basic optimality conditions derived in §F.2 require little modification in handling the extensions.

F.1.2 Background. For all the popularity shown in the literature relating to the derivation of necessary conditions for optimality (see for example the survey of Robinson 1971), it was surprising to find no results directly applicable to the present modelling and design treatment for type I systems. Using a calculus of variations approach, Sage (1968) gives the Euler-Lagrange equations for the problem at hand but does not admit constraints or discontinuities in the controls. Wang and Tung (1964) and Wang (1964) give the equivalent Hamilton-Jacobi-Bellman functional equation formulation and indicate the form of the Hamilton canonical equations where the state function space is a Hilbert space. The proof given here is based on Rozonoer's method. It is without the restricting assumptions of the calculus of variations or the dynamic programming arguments of Sage, Wang and Tung. The generality of the results is also increased over these works.

Numerous uses of the type I form for particular problems in optimal control have been reported. For example, in relation to dynamic programming as a solution technique see Brogan (1967a,b, 1968a, 1968b); for the heat equation Erzberger and Kim (1966a, 1966b), Kim and Erzberger (1967) and Butkovskii (1969); Sage and Chaudhuri (1967) discuss discretization problems; Kim and Gajwani (1968) use the calculus of variations for an integral criterion over time; Katz (1964) discusses both lumped and distributed parameter systems of type I form under a general operational equation using functional analysis arguments; Denn (1966) (see also 1969) and Chaudhuri (1965) use a Green's function approach to derive necessary conditions for linear equations.

The basic method of solution adopted here follows the approach of Rozonoer (1959) for lumped parameter systems. Several other authors have found Rozonoer's approach amenable to an extension to distributed parameter systems. In particular A.I. Egorov has given necessary conditions for quasilinear partial differential equations (1963), hyperbolic, parabolic and elliptic equations (1966), general second order partial differential equations (1964), hyperbolic and parabolic equations (1967a, 1967b). In many of these derivations, local sufficiency is also shown for the linear case. The independent variables need not be fixed in range. The systems are essentially interpreted in a type III format. For systems of a type I form, A.I. Egorov (1965a, 1965b) treats the linear heat conduction equation. Sirazetdinov (1964) considers quasilinear system equations in a simplified type I form with derivatives of state up to the first order on the right hand side. Butkovskii (1969) discusses the work of the last two authors and includes a special control case.

F.2 THE BASIC NECESSARY CONDITIONS.

F.2.1 Preliminary notes and assumptions. Consider a general distributed parameter system described by the system of partial differential equations (system model type I - (C.3.1)),

(f.2.1) $$\frac{\partial x_i}{\partial y_4} = f_i\,[\underline{y},\ \underline{x},\ \ldots\ ,\ \partial_{\underline{\ell}}\underline{x},\ \ldots\ ,\ \underline{u}] \qquad (i = 1,\ \ldots\ ,\ n)$$

where $\underline{\ell} = (\ell_1,\ \ell_2,\ \ell_3)$ and $\partial_{\underline{\ell}}\underline{x}$ is as defined in the 'notation'. $\underline{x}(\underline{y}) = (x_1,\ \ldots\ ,\ x_n)^T$ denotes the state and $\underline{u}(\underline{y}) = (u_1,\ \ldots\ ,\ u_r)^T$ the control at any $\underline{y}=(y_1,\ldots\ ,\ y_4)^T$; $y_i \in [y_i^L,\ y_i^R]$; $i = 1,\ \ldots\ ,\ 4$. The values that the control $\underline{u}$ may take will be assumed to be restricted to a region U in the space of controls with coordinates $u_1,\ \ldots\ ,\ u_r$ (§D.2).

(f.2.2) $$\underline{u}(\underline{y}) \in U$$

The control functions will be assumed to have piecewise continuous properties. Admissible controls will then belong to U and be piecewise continuous.

Boundary conditions are of the form:

$$\text{(f.2.1a)} \qquad \begin{array}{ll} \partial_{(\underline{\ell}-1)} x_i & \text{given at } y_k^L,\ y_k^R\ ,\ k = 1,\ 2,\ 3 \\ x_i & \text{given at } y_4^L,\ y_4^R \end{array}$$

where $\partial_{(\underline{\ell}-1)}\underline{x}$ denotes derivates of the type $\partial_{\underline{\ell}}\underline{x}$ as defined in the 'notation' but to the overall power (L-1) in the numerator and with the power of either y_1, y_2 or y_3 reduced by one in the denominator. The form of the boundary conditions bears a direct relationship to the $\partial_{\underline{\ell}}\underline{x}$ differential terms appearing on the right hand side of the system equations (f.2.1). It is assumed that the boundary conditions (f.2.1a) in association with (f.2.1) define the state for a given control. Conditions (f.2.1a) are intended to imply 'split' conditions at y_i^L and y_i^R where this occurs. The values taken by i in (f.2.1a) are determined by the conditions of any given problem.

The control is to be selected from all the admissible controls so that the functional (§D.3)

$$\text{(f.2.3)} \qquad Q = \int_Y f_o[\underline{y},\ \underline{x},\ \ldots\ ,\ \partial_{\underline{\ell}}\underline{x},\ \ldots\ ,\ \underline{u}]\,d\underline{y}$$

takes on a minimum value. In (f.2.3), $Y \subset E^4$ has coordinates $\underline{y} = (y_1,\ \ldots\ ,y_4)^T$

To facilitate the computations, introduce the auxiliary variable x_o where

$$\text{(f.2.4)} \qquad \begin{array}{l} \dfrac{\partial x_o}{\partial y_4} = \displaystyle\int_{\tilde{Y}} f_o[\underline{y},\ \underline{x},\ \ldots\ ,\ \partial_{\underline{\ell}}\underline{x},\ \ldots\ ,\ \underline{u}]\,d\tilde{y} \\ x_o(y_4^L,\ \tilde{y}) = 0 \end{array}$$

In (f.2.4), $\tilde{Y} \subset E^3$ with coordinates $y = (y_1,\ \ldots\ ,\ y_3)^T$, and if $Y_4 \subset E^1$

with coordinates y_4, then $Y = \underset{\sim}{Y} \times Y_4$. Then the problem reduces to minimizing

(f.2.5) $$Q = x_o(y_4^R, \underset{\sim}{y})$$

To effect a solution of this optimization problem, introduce (n+1) adjoint functions λ_i corresponding to the (n+1) state functions x_i and defined by the relations ('adjoint equations') (see Sage 1968):

(f.2.6) $$\frac{\partial \lambda_i}{\partial y_4} = - \sum_{j=o}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} - (-1)^L \partial_{\underline{\ell}} \left[\sum_{j=o}^{n} \lambda_j \frac{\partial f_j}{\partial [\partial_{\underline{\ell}} x_i]} \right]$$

$(i = 0, 1, \ldots , n)$

with natural boundary conditions

x_i given or

(f.2.6a) $$(-1)^{L-1} \partial_{(\underline{\ell}-1)} \left[\sum_{j=o}^{n} \lambda_j \frac{\partial f_j}{\partial [\partial_{\underline{\ell}} x_i]} \right] = 0 \quad \text{at } y_k^L,\ y_k^R,\ k = 1, 2, 3.$$

$(i = 0, 1, \ldots , n)$

The form of the natural boundary conditions (f.2.6a) and the third term in the adjoint equation (f.2.6) bear a direct relationship to the equivalent $\partial_{\underline{\ell}} \underline{x}$ differential terms that occur on the right hand side of the system equations (f.2.1). There are as many natural boundary condition and adjoint equation third terms as there are $\partial_{\underline{\ell}} \underline{x}$ derivatives of the state in the system equations. Equations (f.2.6) still require the specification of boundary conditions on the interval $[y_4^L, y_4^R]$ before λ_i, $i = 0, 1, \ldots , n$, are completely defined.

Introduce a function H, the Hamiltonian, which is a scalar product of the adjoint variables and the left hand sides of the system equations (f.2.1). Set

(f.2.7) $$H[\underline{y}, \underline{s}, \underline{u}] \triangleq \sum_{i=o}^{n} \lambda_i(\underline{y}) f_i [\underline{y}, \underline{x}, \ldots , \partial_{\underline{\ell}} \underline{x}, \ldots , \underline{u}]$$

where a vector $\underline{s}$ (of ζ components) has been introduced,

$$\underline{s} = (x_o, \; \dots \; , x_n; \; \lambda_o, \; \dots \; , \lambda_n; \; \dots \; , \partial_{\underline{\ell}} x_o, \; \dots \; , \partial_{\underline{\ell}} x_n, \; \dots \;)^T$$

By using the Hamiltonian notation, the system (f.2.1) and adjoint (f.2.6) equations simplify to become

$$\text{(f.2.8)} \qquad \begin{aligned} \frac{\partial x_i}{\partial y_4} &= \frac{\partial H}{\partial \lambda_i} \\ \frac{\partial \lambda_i}{\partial y_4} &= -\frac{\partial H}{\partial x_i} - (-1)^L \partial_{\underline{\ell}} \left[\frac{\partial H}{\partial [\partial_{\underline{\ell}} x_i]} \right] \end{aligned} \qquad \begin{aligned} &(i = 0, 1, \dots , n) \\ &\underline{\ell} = (\ell_1, \ell_2, \ell_3) \end{aligned}$$

It is assumed that the functions f_i, $i = 0, 1, \dots , n$ are continuous in all their arguments and differentiable with respect to the state and control up to the second derivative, in the region Y.

F.2.2 Derivation. The approach of Rozonoer (1959) is to consider a functional (in fact an identity involving the integral of the Hamiltonian over the region Y), the effect of perturbations in this functional, and the relation of this to the perturbations in the criterion away from the optimal solution.

Consider the functional

$$\text{(f.2.9)} \qquad J(\underline{s}, \underline{u}) = \int_Y \left[\sum_{i=o}^{n} \lambda_i \frac{\partial x_i}{\partial y_4} - H \right] d\underline{y} = 0$$

which follows from (f.2.7) and (f.2.8). For some control $\underline{u}(\underline{y}) \in U$, let $\underline{s}(\underline{y})$ be a solution of (f.2.8). If the control is incremented $\underline{\delta u}(\underline{y})$, let the corresponding solution to (f.2.8) be $\underline{s}(\underline{y}) + \underline{\delta s}(\underline{y})$ for the same boundary and terminal conditions. $\underline{\delta u}$ is chosen arbitrary but consistent with the admissibility requirements on $\underline{u}$. The functions $\underline{\delta s}(\underline{y})$ obviously satisfy the equations (containing variations $\underline{\delta u}$, and $\underline{\delta s}$ due to $\underline{\delta u}$);

$$\delta \left(\frac{\partial x_i}{\partial y_4} \right) = \delta \left(\frac{\partial H}{\partial \lambda_i} \right) \qquad \begin{aligned} &(i = 0, 1, \dots , n) \\ &\underline{\ell} = (\ell_1, \ell_2, \ell_3) \end{aligned}$$

(f.2.10)

$$\delta\left(\frac{\partial\lambda_i}{\partial y_4}\right) = -\delta\left(\frac{\partial H}{\partial x_i}\right) - (-1)^L \partial_{\underline{\ell}}\left[\delta\left(\frac{\partial H}{\partial[\partial_{\underline{\ell}} x_i]}\right)\right]$$

with the supplementary conditions

$$\left[\begin{array}{l} \delta x_i = 0 \text{ when } x_i \text{ given or} \\ (-1)^{L-1}\partial_{(\underline{\ell}-1)}\left[\delta\left(\frac{\partial H}{\partial[\partial_{\underline{\ell}} x_i]}\right)\right] = 0 \qquad \text{at } y_k^L,\ y_k^R,\ k = 1, 2, 3. \end{array}\right.$$

(f.2.10a)

$$\left[\begin{array}{l} \partial_{\underline{\ell}-1}(\delta x_i) = 0 \quad \text{when } \partial_{\underline{\ell}-1}\, x_i \text{ given at } y_k^L,\ y_k^R,\ k = 1, 2, 3. \\ \delta x_i = 0 \quad \text{when } x_i \text{ given at } y_4^L,\ y_4^R \end{array}\right.$$

Supplementary conditions (f.2.10a)[1] and (f.2.10a)[2,3] follow from the natural boundary conditions (f.2.6a) and boundary conditions (f.2.1a) respectively. Similar qualifying remarks hold here; in particular there is a direct correspondence with the $\partial_{\underline{\ell}}\underline{x}$ derivative terms that appear in the system equations. Note that nothing may be said as yet on the supplementary conditions for $\delta\lambda_i$, $i = 0, \ldots, n$ at y_4^L, y_4^R . In (f.2.10) and (f.2.10a),

$$\delta\left(\frac{\partial H}{\partial s_i}\right) = \frac{\partial H[\underline{s} + \underline{\delta s}, \underline{u} + \underline{\delta u}, \underline{y}]}{\partial s_i} - \frac{\partial H[\underline{s}, \underline{u}, \underline{y}]}{\partial s_i}$$

For the increment in the control, the corresponding change in J will be

$$\Delta J = J(\underline{s} + \underline{\delta s}, \underline{u} + \underline{\delta u}) - J(\underline{s}, \underline{u}) = 0$$

Using a Taylor's series expansion

(f.2.11)

$$\Delta J = \int_Y \sum_{o}^{n} \left\{\delta\left(\frac{\partial x_i}{\partial y_4}\right)\lambda_i + \delta\lambda_i \frac{\partial x_i}{\partial y_4} + \delta\left(\frac{\partial x_i}{\partial y_4}\right)\delta\lambda_i\right\} d\underline{y}$$

$$- \int_Y \left\{H[\underline{s} + \underline{\delta s}, \underline{u} + \underline{\delta u}, \underline{y}] - H[\underline{s}, \underline{u}, \underline{y}]\right\} d\underline{y}$$

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Integrate term (1) of (f.2.11) over y_4 and substituting for $\frac{\partial \lambda_i}{\partial y_4}$ from (f.2.8)

$$\int_Y \sum_o^n \lambda_i \delta \left(\frac{\partial x_i}{\partial y_4} \right) d\underline{y}$$

$$= \int_{\underset{\sim}{Y}} \sum_o^n \left\{ \lambda_i \delta x_i \Big|_{y_4^L}^{y_4^R} + \int_{Y_4} \delta x_i \left(\frac{\partial H}{\partial x_i} + (-1)^L \partial_{\underline{\ell}} \left[\frac{\partial H}{\partial [\partial_{\underline{\ell}} x_i]} \right] \right) dy_4 \right\} d\underset{\sim}{y}$$

Integrate the last term by parts over $\underset{\sim}{Y}$ and using the natural boundary conditions (f.2.6a)

$$\text{(f.2.12)} \quad = \int_{\underset{\sim}{Y}} \sum_o^n \left\{ \lambda_i \delta x_i \Big|_{y_4^L}^{y_4^R} + \int_{Y_4} \left[\delta x_i \frac{\partial H}{\partial x_i} + \delta(\partial_{\underline{\ell}} x_i) \frac{\partial H}{\partial [\partial_{\underline{\ell}} x_i]} \right] dy_4 \right\} d\underset{\sim}{y}$$

Term (2) of (f.2.11) by direct substitution for $\frac{\partial x_i}{\partial y_4}$ from (f.2.8) may be written

$$\text{(f.2.13)} \quad \int_Y \sum_o^n \delta\lambda_i \frac{\partial x_i}{\partial y_4} d\underline{y} = \int_Y \sum_o^n \delta\lambda_i \frac{\partial H}{\partial \lambda_i} d\underline{y}$$

Integrate term (3) of (f.2.11) by parts over Y_4 and substituting for $\delta \left(\frac{\partial \lambda_i}{\partial y_4} \right)$ from (f.2.10),

$$\int_Y \sum_o^n \delta\lambda_i \, \delta \left(\frac{\partial x_i}{\partial y_4} \right) d\underline{y}$$

$$= \int_{\underset{\sim}{Y}} \sum_o^n \left\{ \delta\lambda_i \delta x_i \Big|_{y_4^L}^{y_4^R} \right.$$

$$\left. + \int_{Y_4} \delta x_i \left[\delta \left(\frac{\partial H}{\partial x_i} \right) + (-1)^L \partial_{\underline{\ell}} \left[\delta \left(\frac{\partial H}{\partial [\partial_{\underline{\ell}} x_i]} \right) \right] \right] dy_4 \right\} d\underset{\sim}{y}$$

Integrating the last term by parts over $\underset{\sim}{Y}$ and using the supplementary conditions (f.2.10a)

$$= \int_{\underset{\sim}{Y}} \sum_{o}^{n} \left\{ \delta\lambda_i \delta x_i \Big|_{y_4^L}^{y_4^R} \right.$$

$$\left. + \int_{Y_4} \left[\delta\left(\frac{\partial H}{\partial x_i}\right) \delta x_i + \delta\left(\frac{\partial H}{\partial [\partial_{\underline{\ell}} x_i]}\right) \delta(\partial_{\underline{\ell}} x_i) \right] dy_4 \right\} d\underset{\sim}{y}$$

Also term (3) of (f.2.11) using (f.2.10) may be expressed as

$$\int_Y \sum_o^n \delta\lambda_i \; \delta\left(\frac{\partial x_i}{\partial y_4}\right) d\underline{y} = \int_Y \sum_o^n \delta\lambda_i \; \delta\left(\frac{\partial H}{\partial \lambda_i}\right) d\underline{y}$$

Combining the two reductions of term (3) of (f.2.11), that is the last two lines

(f.2.14) $$\int_Y \sum_o^n \delta\lambda_i \; \delta\left(\frac{\partial x_i}{\partial y_4}\right) d\underline{y} = \frac{1}{2} \int_Y \sum_1^{\zeta} \delta s_i \; \delta\left(\frac{\partial H}{\partial s_i}\right) d\underline{y} + \frac{1}{2} \int_{\underset{\sim}{Y}} \sum_o^n \delta\lambda_i \delta x_i \Big|_{y_4^L}^{y_4^R} d\underset{\sim}{y}$$

where the upper summation limit equals the number of components in the vector $\underline{s}$.

Terms (4) and (5) of (f.2.11) may be expanded in a finite Taylor's series

(f.2.15) $$\int_Y \left\{ H[\underline{s} + \underline{\delta s}, \underline{u} + \underline{\delta u}, \underline{y}] - H[\underline{s}, \underline{u}, \underline{y}] \right\} d\underline{y}$$

$$= \int_Y \left\{ H[\underline{s}, \underline{u} + \underline{\delta u}, \underline{y}] - H[\underline{s}, \underline{u}, \underline{y}] \right.$$

$$+ \sum_1^{\zeta} \delta s_i \frac{\partial}{\partial s_i} (H[\underline{s}, \underline{u} + \underline{\delta u}, \underline{y}])$$

$$\left. + \sum_{i,j=1}^{\zeta} \frac{\delta s_i \delta s_j}{2} \frac{\partial^2}{\partial s_i \partial s_j} (H[\underline{s} + \theta_1 \underline{\delta s}, \underline{u} + \underline{\delta u}, \underline{y}]) \right\} d\underline{y}$$

where $0 < \theta_1(\underline{y}) < 1$.

Using (f.2.12), ... , (f.2.15), ΔJ in (f.2.11) then equals, after rearranging terms

$$\Delta J = \int_{\underset{\sim}{Y}} \sum_{o}^{n} \lambda_i \delta x_i \Big|_{y_4^L}^{y_4^R} d\underset{\sim}{y} - \int_{Y} \Big\{ H[\underline{s}, \underline{u} + \underline{\delta u}, \underline{y}] - H[\underline{s}, \underline{u}, \underline{y}] \Big\} d\underline{y}$$

$$- \int_{Y} \sum_{i,j=1}^{\zeta} \frac{\delta s_i \delta s_j}{2} \frac{\partial^2}{\partial s_i \partial s_j} (H[\underline{s} + \theta_1 \underline{\delta s}, \underline{u} + \underline{\delta u}, \underline{y}]) d\underline{y}$$

$$- \frac{1}{2} \int_{Y} \sum_{1}^{\zeta} \delta s_i \Big\{ \frac{\partial}{\partial s_i} (H[\underline{s}, \underline{u} + \underline{\delta u}, \underline{y}]) - \frac{\partial}{\partial s_i} (H[\underline{s}, \underline{u}, \underline{y}]) \Big\} d\underline{y}$$

$$+ \frac{1}{2} \int_{Y} \sum_{1}^{\zeta} \delta s_i \Big\{ \frac{\partial}{\partial s_i} (H[\underline{s} + \underline{\delta s}, \underline{u} + \underline{\delta u}, \underline{y}]) - \frac{\partial}{\partial s_i} (H[\underline{s}, \underline{u} + \underline{\delta u}, \underline{y}]) \Big\} d\underline{y}$$

$$+ \frac{1}{2} \int_{\underset{\sim}{Y}} \sum_{o}^{n} \delta \lambda_i \delta x_i \Big|_{y_4^L}^{y_4^R} d\underset{\sim}{y}$$

Using a finite Taylor's series for the second last term and knowing that $\Delta J = 0$, then

$$\text{(f.2.16)} \qquad \int_{\underset{\sim}{Y}} \sum_{o}^{n} \lambda_i \delta x_i \Big|_{y_4^L}^{y_4^R} d\underset{\sim}{y} = \int_{Y} \Big\{ H[\underline{s}, \underline{u} + \underline{\delta u}, \underline{y}] - H[\underline{s}, \underline{u}, \underline{y}] \Big\} d\underline{y} + \eta$$

where $\eta = \eta_1 + \eta_2 + \eta_3$ and

$$\eta_1 = \frac{1}{2} \int_{Y} \sum_{1}^{\zeta} \delta s_i \Big\{ \frac{\partial}{\partial s_i} (H[\underline{s}, \underline{u} + \underline{\delta u}, \underline{y}]) - \frac{\partial}{\partial s_i} (H[\underline{s}, \underline{u}, \underline{y}]) \Big\} d\underline{y}$$

$$\eta_2 = \frac{1}{2} \int_{Y} \sum_{i,j=1}^{\zeta} \delta s_i \delta s_j \Big\{ \frac{\partial^2 H[\underline{s} + \theta_1 \underline{\delta s}, \underline{u} + \underline{\delta u}, \underline{y}]}{\partial s_i \partial s_j}$$

$$- \frac{\partial^2 H[\underline{s} + \theta_2 \underline{\delta s}, \underline{u} + \underline{\delta u}, \underline{y}]}{\partial s_i \partial s_j} \Big\} d\underline{y}$$

$$\eta_3 = - \frac{1}{2} \int_{\underset{\sim}{Y}} \sum_{o}^{n} \delta\lambda_i \delta x_i \Big|_{y_4^L}^{y_4^R} d\underset{\sim}{y}$$

and $0 < \theta_2(\underline{y}) < 1$.

Before obtaining estimates of the terms in equation (f.2.16), return for the moment to the characterization of the adjoint variables λ_i, $i = 0, 1, \ldots, n$; for a satisfactory definition of the adjoint, suitable end conditions over the interval $[y_4^L, y_4^R]$ have to be associated with equation (f.2.6). Also by-passing this question for the moment and assuming satisfactory end conditions exist, then the solution of (f.2.6) has the property that its scalar product with the state increment is constant over the interval $[y_4^L, y_4^R]$. Formally

$$\text{(f.2.17)} \qquad \int_{\underset{\sim}{Y}} \sum_{o}^{n} \delta x_i \lambda_i \, d\underset{\sim}{y} = \text{constant} \qquad \forall y_4 \in [y_4^L, y_4^R]$$

This may be verified by differentiation. In particular

$$\int_{\underset{\sim}{Y}} \frac{\partial}{\partial y_4} \left(\sum_{o}^{n} \delta x_i \lambda_i \right) d\underset{\sim}{y}$$

$$= \int_{\underset{\sim}{Y}} \left(\sum_{i=o}^{n} \delta x_i \frac{\partial \lambda_i}{\partial y_4} + \sum_{j=o}^{n} \lambda_j \frac{\partial(\delta x_j)}{\partial y_4} \right) d\underset{\sim}{y}$$

Now $\frac{\partial(\delta x_j)}{\partial y_4}$ may be obtained from the 'variational equations' for $x_j(\underline{y})$, derived by considering arbitrarily small variations $\delta x_j(\underline{y})$ from $x_j(\underline{y})$. From (f.2.1) and (f.2.4)

$$\frac{\partial x_j}{\partial y_4} = f_j[\underline{y}, \underline{x}, \ldots, \partial_{\ell}\underline{x}, \ldots, \underline{u}] \qquad j = 0, 1, \ldots, n$$

Replacing x_j with $x_j + \delta x_j$ and expanding f_j in a Taylor's series about its unperturbed solution,

$$\frac{\partial(x_j + \delta x_j)}{\partial y_4} = f_j\,[\underline{y},\ \underline{x} + \underline{\delta x},\ \ldots\ ,\ \partial_{\underline{\ell}}(\underline{x} + \underline{\delta x}),\ \ldots\ ,\ \underline{u}]$$

$$j = 0,\ 1,\ \ldots\ ,\ n$$

$$= f_j\,[\underline{y},\ \underline{x},\ \ldots\ ,\ \partial_{\underline{\ell}}\underline{x},\ \ldots\ ,\ \underline{u}]$$

$$+ \sum_{i=o}^{n} \delta x_i\ \frac{\partial f_j\,[\ \underline{y},\ \underline{x},\ \ldots\ ,\ \partial_{\underline{\ell}}\underline{x},\ \ldots\ ,\ \underline{u}]}{\partial x_i}$$

$$+ \sum_{i=o}^{n} \delta(\partial_{\underline{\ell}} x_i)\ \frac{\partial f_j}{\partial\,[\partial_{\underline{\ell}} x_i]}\ [\ \underline{y},\ \underline{x},\ \ldots\ ,\ \partial_{\underline{\ell}}\underline{x},\ \ldots\ ,\ \underline{u}]$$

$$+ \ \ldots$$

Taking (f.2.1) and (f.2.4) into account, the (linear) variational equations result;

$$\text{(f.2.18)} \qquad \frac{\partial(\delta x_j)}{\partial y_4} = \sum_{i=o}^{n} \delta x_i\ \frac{\partial f_j}{\partial x_i} + \sum_{i=o}^{n} \delta(\partial_{\underline{\ell}} x_i) \frac{\partial f_j}{\partial\,[\partial_{\underline{\ell}} x_i]}$$

$$j = 0,\ 1,\ \ldots\ ,\ n$$

This substitution for $\dfrac{\partial(\delta x_j)}{\partial y_4}$ may now be made;

$$\int_{\underset{\sim}{Y}} \frac{\partial}{\partial y_4}\left(\sum_{o}^{n} \delta x_i \lambda_i\right)\, d\underset{\sim}{y}$$

$$= \int_{\underset{\sim}{Y}} \left(\sum_{i=o}^{n} \delta x_i\ \frac{\partial \lambda_i}{\partial y_4} + \sum_{j=o}^{n} \lambda_j \left[\sum_{i=o}^{n} \delta x_i\ \frac{\partial f_j}{\partial x_i} + \sum_{i=o}^{n} \delta(\partial_{\underline{\ell}} x_i)\ \frac{\partial f_j}{\partial\,[\partial_{\underline{\ell}} x_i]}\right]\right)\, d\underset{\sim}{y}$$

Interchanging the order of summations

$$= \int_{\underset{\sim}{Y}} \left(\sum_{i=o}^{n} \delta x_i \frac{\partial \lambda_i}{\partial y_4} + \sum_{i=o}^{n} \delta x_i \sum_{j=o}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} \right.$$

$$\left. + \sum_{i=o}^{n} \delta (\partial_{\underline{\ell}} x_i) \sum_{j=o}^{n} \lambda_j \frac{\partial f_j}{\partial [\partial_{\underline{\ell}} x_i]} \right) d\underset{\sim}{y}$$

The last term may be expanded using the product rule repeatedly,

$$= \int_{\underset{\sim}{Y}} \left(\sum_{i=o}^{n} \delta x_i \frac{\partial \lambda_i}{\partial y_4} + \sum_{i=o}^{n} \delta x_i \left\{ \sum_{j=o}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} + (-1)^{L} \partial_{\underline{\ell}} \left[\sum_{j=o}^{n} \lambda_j \frac{\partial f_j}{\partial [\partial_{\underline{\ell}} x_i]} \right] \right\} \right.$$

$$+ \sum_{k=1}^{3} \frac{\partial}{\partial y_k} \left\{ \sum_{i=o}^{n} \partial_{(\underline{\ell}-1)} (\delta x_i) \sum_{j=o}^{n} \lambda_j \frac{\partial f_j}{\partial [\partial_{\underline{\ell}} x_i]} \right\}$$

$$- \sum_{k=1}^{3} \frac{\partial}{\partial y_k} \left\{ \sum_{i=o}^{n} \partial_{(\underline{\ell}-2)} (\delta x_i)\, \partial_1 \sum_{j=o}^{n} \lambda_j \frac{\partial f_j}{\partial [\partial_{\underline{\ell}} x_i]} \right\}$$

$$+ \sum_{k=1}^{3} \frac{\partial}{\partial y_k} \left\{ \sum_{i=o}^{n} \partial_{(\underline{\ell}-3)} (\delta x_i)\, \partial_2 \sum_{j=o}^{n} \lambda_j \frac{\partial f_j}{\partial [\partial_{\underline{\ell}} x_i]} \right\}$$

$$\left. - \ldots + (-1)^{L-1} \sum_{k=1}^{3} \frac{\partial}{\partial y_k} \left\{ \sum_{i=o}^{n} \delta x_i\, \partial_{(\underline{\ell}-1)} \sum_{j=o}^{n} \lambda_j \frac{\partial f_j}{\partial [\partial_{\underline{\ell}} x_i]} \right\} \right) d\underset{\sim}{y}$$

The first three terms cancel, following equation (f.2.6). After integrating over $\underset{\sim}{Y}$ the last term, from (f.2.6a), vanishes. After integrating over $\underset{\sim}{Y}$, the remaining terms are set equal to zero; of these remaining terms some will be inherently zero from the boundary conditions (f.2.1a)[1] while the optimal solution will be required to be of a form such that the other remaining terms are zero.

Hence

$$(f.2.19) \qquad \int_{\underset{\sim}{Y}} \sum_{o}^{n} \delta x_i \lambda_i d\underset{\sim}{y} = \int_{\underset{\sim}{Y}} \sum_{o}^{n} \delta x_i \lambda_i \Big|_{y_4^L} d\underset{\sim}{y} = \int_{\underset{\sim}{Y}} \sum_{o}^{n} \delta x_i \lambda_i \Big|_{y_4^R} d\underset{\sim}{y} = \text{constant}$$

$$\forall y_4 \in [y_4^L, y_4^R]$$

With a desired result in mind, a suitable choice of end conditions over the interval $[y_4^L, y_4^R]$ on the adjoint variables λ_i, $i = 0, \ldots, n$ for equation (f.2.6) would be such that the increment in the criterion (due to the increment in control $\underset{\sim}{\delta u}$) equalled this constant. (It will be recalled that the choice of end conditions over the interval $[y_4^L, y_4^R]$ on λ_i was completely free.) That is

$$(f.2.20) \qquad -\delta Q = -\int_{\underset{\sim}{Y}} \delta x_o \Big|_{y_4^R} d\underset{\sim}{y} = \int_{\underset{\sim}{Y}} \sum_{o}^{n} \delta x_i \lambda_i \Big|_{y_4^L,\, y_4^R} d\underset{\sim}{y} = \int_{\underset{\sim}{Y}} \sum_{o}^{n} \delta x_i \lambda_i \, d\underset{\sim}{y} \leq 0$$

This implies that at any y_4, the scalar product of λ_i and δx_i is chosen to be a maximum (and vanishes at the optimum). As a special case, at y_4^R the scalar product is chosen equal to $\int_{\underset{\sim}{Y}} \delta x_o d\underset{\sim}{y}$; evidently $\lambda_o(y_4^R, \underset{\sim}{y}) = -1$ and the remaining λ_i, $i = 1, \ldots, n$ are chosen so that the scalar product of λ_i and δx_i, $i = 1, \ldots, n$ equals zero. At y_4^L the scalar product of λ_i and δx_i, $i = 0, 1, \ldots, n$ is chosen to be zero for the optimum.

In general, only n end conditions on $\underset{\sim}{x}$ will be specified over Y_4 for a meaningful problem, and these will be shared between $y_4 = y_4^L$ and $y_4 = y_4^R$. (f.2.20) implies that for $i = 1, \ldots, n$, $\lambda_i(y_4^L, \underset{\sim}{y}) = 0$ when $x_i(y_4^L, \underset{\sim}{y})$ is free to vary and $\lambda_i(y_4^L, \underset{\sim}{y})$ is free to vary whenever $x_i(y_4^L, \underset{\sim}{y})$ is specified. (Note, $x_o(y_4^L, \underset{\sim}{y})$ is specified zero by definition and hence $\delta x_o(y_4^L, \underset{\sim}{y}) = 0$ and $\lambda_o(y_4^L, \underset{\sim}{y})$ is free.) Similar implications hold at $y_4 = y_4^R$.

(Having found the characterization of the boundary conditions on λ_i at y_4^L, y_4^R, the supplementary conditions of equations (f.2.10a) may now be completed. As is applicable with the state, $\delta\lambda_i = 0$ at y_4^L, y_4^R when λ_i is known from the above natural boundary conditions resulting from (f.2.20). Obviously under these circumstances $\eta_3 = 0$.)

Returning now to equation (f.2.16)

$$\int_{\underset{\sim}{Y}} \sum_{0}^{n} \lambda_i \delta x_i \Big|_{y_4^L}^{y_4^R} d\underset{\sim}{y} = -\delta Q$$

and the increment in the criterion

$$\text{(f.2.21)} \qquad \delta Q = -\int_{Y} \left\{ H[\underline{s}, \underline{u} + \underline{\delta u}, \underline{y}] - H[\underline{s}, \underline{u}, \underline{y}] \right\} d\underline{y} - \eta$$

where now $\eta = \eta_1 + \eta_2$ only.

The expression for δQ is now in the same form arrived at by Rozonoer (1959), Sirazetdinov (1964), Butkovskii (1969) and A.I. Egorov in his many papers. These authors, following Rozonoer, establish estimates for the remainder term η in terms of the increment in control $\underline{\delta u}$ and then use an argument in contradiction to show that the control is minimizing when it satisfies a 'maximum condition'

$$\text{(f.2.22)} \qquad H[\underline{s}, \underline{\hat{u}}, \underline{y}] \geq H[\underline{s}, \underline{u}, \underline{y}] \qquad\qquad \underline{u} \in U$$

That is, the Hamiltonian H reaches its maximum in the region U for optimum $\underline{u}$. (See the above listed references and in particular Rozonoer.)

However the establishment of an estimate for the remainder term η appears difficult for the general nonlinear case considered here, requiring knowledge of particular results in the theory of inequalities. When (f.2.1) is quasilinear, the result (f.2.22) follows from the work of Sirazetdinov, while for general nonlinear system equations of type III form, the result (f.2.22) follows from the work of A.I. Egorov (1964) and Butkovskii. It would then appear reasonable that with the necessary mathematical expertise, the result (f.2.22) would follow for the general nonlinear type I form. An alternative, and simpler, approach using physical arguments will however be used here to establish this maximum condition. The approach originates from the lumped parameter work of Fel'dbaum (1965). Whereas the complete derivation of the estimate for η in Rozonoer's method is pure mathematics, the following route to the 'maximum condition' will enable a 'feel' for what is happening in the derivation.

In particular, consider a small localised perturbation in the control (but this time perturbed from the optimal $\hat{u}$ which is assumed known) during the infinitesimally small interval $(\gamma-\varepsilon,\gamma)$ for $\gamma \in [y_4^L, y_4^R]$ and ε small. This is consistent with the piecewise continuous assumptions on $\underline{u}$. Outside of this interval, the control assumes a value $\underline{u}$ say, $\underline{u} \in U$, $\underline{u} \neq \underline{\hat{u}}$.

From physical arguments, this variation in the control has only a small influence on the system for $y_4 > \gamma$. The subsequent variation in the state is

$$\underline{\underline{x}}(y_4, \underset{\sim}{y}) - \underline{\underline{\hat{x}}}(y_4, \underset{\sim}{y}) = \underline{\underline{\delta x}}(y_4, \underset{\sim}{y})$$

where the double underscore indicates (n+1) component vectors.

For $y_4 = \gamma$, to an accuracy commensurate with that adopted in the derivation of the variational equations (f.2.18) - that is neglecting higher order small terms in δx of the same magnitude neglected in obtaining (f.2.18);

$$\underline{\underline{\delta x}}(\gamma, \underset{\sim}{y}) = \varepsilon\left(\left[\frac{\partial \underline{\underline{x}}(y_4,\underset{\sim}{y})}{\partial y_4}\right]_{y_4 = \gamma} - \left[\frac{\partial \underline{\underline{\hat{x}}}(y_4,\underset{\sim}{y})}{\partial y_4}\right]_{y_4 = \gamma}\right)$$

$$= \varepsilon\Big[\underline{f}[\gamma, \underset{\sim}{y}, \underline{\underline{x}}, \ldots , \partial_{\underline{\ell}}\underline{\underline{x}}, \ldots , \underline{u}]$$

$$- \underline{f}[\gamma, \underset{\sim}{y}, \underline{\underline{x}}, \ldots , \partial_{\underline{\ell}}\underline{\underline{x}}, \ldots , \underline{\hat{u}}]\Big]$$

Using (f.2.20)

$$\int_{\underset{\sim}{Y}}\left\{ \sum_{o}^{n}(f_i[\gamma, \underset{\sim}{y}, \underline{\underline{x}}, \ldots , \partial_{\underline{\ell}}\underline{\underline{x}}, \ldots , \underline{u}]\lambda_i - f_i[\gamma, \underset{\sim}{y}, \underline{\underline{x}}, \ldots ,\right.$$

$$\left. \partial_{\underline{\ell}}\underline{\underline{x}}, \ldots , \underline{\hat{u}}]\lambda_i)\right\} d\underset{\sim}{y} \leq 0$$

This holds for all $y_4 = \gamma \in [y_4^L, y_4^R]$. In terms of the Hamiltonian defined in (f.2.7)

$$\int_{\underset{\sim}{Y}} H[\underline{y}, \underline{s}, \underline{u}] - H[\underline{y}, \underline{s}, \hat{\underline{u}}] \, d\underset{\sim}{y} \leq 0$$

That is H attains its maximum value for the optimal control $\hat{u}(\underline{y})$. That is $\underline{u}(\underline{y})$ is chosen to maximise the Hamiltonian. This is the required 'maximum condition.'

F.2.3 Summary of the conditions: If for a system described by a set of partial differential equations (state, system equations) of the form

$$\text{(f.2.1)} \qquad \frac{\partial x_i}{\partial y_4} = f_i(\underline{y}, \underline{x}, \ldots, \partial_{\underline{\ell}}\underline{x}, \ldots, \underline{u}) \qquad i = 1, \ldots, n$$

with

$$\text{(f.2.1a)} \qquad \begin{array}{ll} \partial_{(\underline{\ell}-1)} x_i & \text{given at } y_k^L, y_k^R, \quad k = 1, 2, 3 \\ x_i & \text{given at } y_4^L, y_4^R \end{array}$$

it is required that the criterion

$$\text{(f.2.3)} \qquad Q \mathrel{\hat{=}} x_o = \int_Y f_o(\underline{y}, \underline{x}, \ldots, \partial_{\underline{\ell}}\underline{x}, \ldots, \underline{u}) \, d\underline{y}$$

be minimized, (n+1) auxiliary (adjoint) variables $\lambda_i (i = 0, 1, \ldots, n)$ are introduced, defined by the set of equations (adjoint equations)

$$\text{(f.2.8)}^2 \qquad \frac{\partial \lambda_i}{\partial y_4} = -\frac{\partial H}{\partial x_i} - (-1)^L \partial_{\underline{\ell}} \left[\frac{\partial H}{\partial [\partial_{\underline{\ell}} x_i]}\right] \qquad (i = 0, 1, \ldots, n)$$

with natural boundary conditions

$$\text{(f.2.6a)} \qquad x_i \text{ given or } (-1)^{L-1} \partial_{(\underline{\ell}-1)} \left[\frac{\partial H}{\partial [\partial_{\underline{\ell}} x_i]}\right] = 0 \text{ at } y_k^L, y_k^R, \; k = 1, 2, 3$$

(f.2.20) λ_i free (zero) if x_i is given (free), at y_4^L, y_4^R

(as a special case $\lambda_o \Big|_{y_4^R} = -1$)

where the Hamiltonian

$$H = \sum_{o}^{n} \lambda_i(\underline{y})\ f_i(\underline{y}, \underline{x}, \ldots, \underline{\partial_\ell x}, \ldots, \underline{u}) \tag{f.2.7}$$

(Evidently λ_o = constant, -1.)

Then for an optimum system, it is necessary that the Hamiltonian be maximized over all admissible controls. Formally

$$H[\underline{s}, \underline{\hat{u}}, \underline{y}] \geq H[\underline{s}, \underline{u}, \underline{y}] \qquad \underline{u} \in U \tag{f.2.22}$$

where $\underline{s} = (\underline{x}, \underline{\lambda}, \ldots, \underline{\partial_\ell x}, \ldots)^T$

This last inequality (f.2.22) will be variously referred to in the following as the 'optimality condition' or 'control (in)equation', as it is used to obtain a relation for $\underline{\hat{u}}$ in terms of $\underline{s}$ and $\underline{y}$.

F.3 SOME EXTENSIONS OF THE NECESSARY CONDITONS.

F.3.1 Introduction. The necessary conditions summarized in the previous article may be given more general applicability, so as to cover a broad class of problems, by considering several extensions. (All possible ramifications are not considered; for example variable independent variable limits, various constraint formulations and others.) It is shown in this article that the necessary conditions of §F.2.3 may be used as a common basis for superficially different problems. The different problems are shown here to require only analogous (and not separate) treatment. Transversality conditions replace the natural boundary conditions on the adjoint variables where end-state constraints or end criteria exist. The remainder of the necessary conditions are unchanged. In this sense, transversality conditions are a partial constraint on the adjoint variables and reflect a corresponding constraint on the design problem.

F.3.2 End-state criterion. In the basic derivation, a criterion defined over the whole Y domain was considered. Consider now a criterion of the form (§D.3)

$$(f.3.1) \qquad Q^* = \int_{\underset{\sim}{Y}} g(\underline{y}, \underline{x}(\underline{y}))\Big|_{y_4^R} d\underset{\sim}{y}$$

An analogous form may be chosen for $y_4 = y_4^L$.

The new problem may be thought of in either of two ways. Firstly, if g is differentiable, then with a treatment similar to the previous derivation, a new variable may be introduced;

$$(f.3.2) \qquad x_o(\underline{y}) \triangleq Q^* \; ; \; x_o(y_4^L, \underset{\sim}{y}) = 0$$

and add to the system equations (f.2.1);

$$(f.3.3) \qquad \frac{\partial x_o}{\partial y_4} = \sum_{i=1}^{n} \frac{\partial Q^*}{\partial x_i} f_i\,[\underline{y}, \underline{x}, \ldots , \partial_{\underline{\ell}}\underline{x}, \ldots , \underline{u}]$$

The problem of minimizing the integral (f.3.1) is then equivalent to minimizing $x_o(y_4^R, \underset{\sim}{y})$ as in (f.2.5) and the whole derivation may be repeated verbatim.

An alternative way of looking at the problem is to examine the change in Q* for a change in the state. If $\underline{x}$ is incremented by $\underline{\delta x}$

$$g(\underline{y}, \underline{x})\Big|_{y_4^R} \to g(\underline{y}, \underline{x} + \underline{\delta x})\Big|_{y_4^R}$$

Expanding in a Taylor's series about $g(\underline{y}, \underline{x})\Big|_{y_4^R}$;

$$g(\underline{y}, \underline{x} + \underline{\delta x})\Big|_{y_4^R} = g(\underline{y}, \underline{x})\Big|_{y_4^R} + \sum_{1}^{n} \delta x_i \frac{\partial g(\underline{y}, \underline{x})}{\partial x_i}\Big|_{y_4^R} + \ldots$$

That is, the change in the criterion, neglecting small order terms

$$\delta Q^* = \int_{\underset{\sim}{Y}} \sum_{1}^{n} \delta x_i \left. \frac{\partial g(\underline{y}, \underline{x})}{\partial x_i} \right|_{y_4^R} d\underset{\sim}{y}$$

Using a similar argument to that in the previous article, a satisfactory choice of the boundary conditions for the adjoint variables is

$$\lambda_i(\underline{y}) = -\frac{\partial g}{\partial x_i} \quad \text{at} \quad y_4^R \qquad\qquad i = 1, \ldots, n$$

(Analogous statements hold for y_4^L.) The remainder of the maximum principle statement is unchanged.

<u>F.3.3 Generalised end states.</u> More general boundary conditions may be postulated than those considered in the basic derivation (§F.2) where the state took specific values.

Consider a general function of the end states (§C.3)

$$\text{(f.3.4)} \qquad \int_{\underset{\sim}{Y}} \underline{S}(\underline{x}(\underline{y}), \ldots, \partial_{\underline{\ell}}\underline{x}(\underline{y}), \ldots) \Big|_{y_4^R} d\underset{\sim}{y} = 0 \quad ; \quad \underline{S} = (S_1, \ldots, S_q)^T$$

A similar function may be specified for the states at y_4^L.

To deal with this situation, (f.3.4) is joined to the criterion by means of Lagrange multipliers $\Lambda_i(\underset{\sim}{y})$, $i = 1, \ldots, q$ in the manner shown;

$$\text{(f.3.5)} \qquad Q = \int_{\underset{\sim}{Y}} g d\underset{\sim}{y} \Big|_{y_4^R} + \int_{Y} f_o d\underline{y} + \int_{\underset{\sim}{Y}} \sum_{1}^{q} \Lambda_i S_i(\underline{x}, \ldots, \partial_{\underline{\ell}}\underline{x}, \ldots) \Big|_{y_4^R} d\underset{\sim}{y}$$

For variations in the state, isolating the last term

$$\delta Q = \ldots \int_{\underset{\sim}{Y}} \sum_{j=1}^{n} \delta x_j \sum_{i=1}^{q} \frac{\partial S_i}{\partial x_j} \Lambda_i$$

$$+ \sum_{j=1}^{n} \delta(\partial_{\underline{\ell}} x_j) \sum_{i=1}^{q} \frac{\partial S_i}{\partial [\partial_{\underline{\ell}} x_j]} \Lambda_i \Big|_{y_4^R} d\underset{\sim}{y}$$

The last term of this line may be rewritten as

$$\delta(\partial_{\underline{\ell}} x_j) \sum_{i=1}^{q} \frac{\partial s_i}{\partial[\partial_{\underline{\ell}} x_j]} \Lambda_i = (\partial_{\underline{\ell}} \delta x_j) \sum_{i=1}^{q} \frac{\partial s_i}{\partial[\partial_{\underline{\ell}} x_j]} \Lambda_i$$

$$= \partial_{\underline{\ell}}^T \left\{ \partial_{\underline{\ell}-1} (\delta x_j) \sum_{i=1}^{q} \frac{\partial s_i}{\partial[\partial_{\underline{\ell}} x_j]} \Lambda_i \right\}$$

$$- \partial_{\underline{\ell}}^T \left\{ \partial_{\underline{\ell}-2} (\delta x_j) \; \partial_1 \left[\sum_{i=1}^{q} \frac{\partial s_i}{\partial[\partial_{\underline{\ell}} x_j]} \Lambda_i \right] \right\}$$

$$+ \partial_{\underline{\ell}}^T \left\{ \partial_{\underline{\ell}-3} (\delta x_j) \; \partial_2 \left[\sum_{i=1}^{q} \frac{\partial s_i}{\partial[\partial_{\underline{\ell}} x_j]} \Lambda_i \right] \right\}$$

$$- \dots + (-1)^{L-1} \partial_{\underline{\ell}}^T \left\{ (\delta x_j) \; \partial_{\underline{\ell}-1} \left[\sum_{i=1}^{q} \frac{\partial s_i}{\partial[\partial_{\underline{\ell}} x_j]} \Lambda_i \right] \right\}$$

$$+ (-1)^{L} \left\{ (\delta x_j) \; \partial_{\underline{\ell}} \left[\sum_{i=1}^{q} \frac{\partial s_i}{\partial[\partial_{\underline{\ell}} x_j]} \Lambda_i \right] \right\}$$

When integrated over $\underset{\sim}{Y}$, the terms preceding the ellipsis dots are identically zero from the same arguments used in obtaining (f.2.19), and the term following the ellipsis dots is also identically zero from the natural boundary conditions on the adjoint variables (expression (f.2.6a)).

The first variation in Q then becomes

$$\delta Q = \dots \int_{\underset{\sim}{Y}} \sum_{j=1}^{n} \delta x_j \left[\sum_{i=1}^{q} \frac{\partial s_i}{\partial x_j} \Lambda_i + (-1)^{L} \partial_{\underline{\ell}} \left[\sum_{i=1}^{q} \frac{\partial s_i}{\partial[\partial_{\underline{\ell}} x_j]} \Lambda_i \right] \right] \Bigg|_{Y_4^R} d\underset{\sim}{y}$$

Again using a similar argument in relation to the boundary conditions on

λ_j, j = 1, ... , n, a suitable choice of λ_j at y_4^R is

$$\text{(f.3.6)} \qquad \lambda_j = - \sum_{i=1}^{q} \frac{\partial s_i}{\partial x_j} \Lambda_i - (-1)^L \partial_{\underline{\ell}} \left\{ \sum_{i=1}^{q} \frac{\partial s_i}{\partial [\partial_{\underline{\ell}} x_j]} \Lambda_i \right\} \Bigg|_{y_4^R}$$

where the q multipliers Λ_i (i = 1, ... , q) are chosen to satisfy (f.3.4).

Similar statements hold for y_4^L. The maximum principle is little changed. These transversality conditions (f.3.6) substitute for the natural boundary conditions of the basic derivation (§F.2). Note that the use of the multipliers $\underline{\Lambda}$ in the above has an analogy in the treatment of isoperimetric (or integral) constraints in the calculus of variations.

F.4 DISCUSSION.

The approach to optimum design, based on the maximum principle, does not give an explicit expression for the optimal control but instead optimality is manifested in a set of necessary conditions that have to be satisfied. The necessary conditions will in general constitute a system of partial differential equations of the boundary value type (end conditions are split), the solution of which may present certain complications. To determine the optimal control and the corresponding state, two sets of equations - the state (f.2.1,8^1) and the adjoint (f.2.6,8^2) are solved simultaneously for 2n + r unknowns $\{x_i;\ i = 1, \ldots, n\}$, $\{\lambda_j;\ j = 1, \ldots, n\}$ and $\{u_k;\ k = 1, \ldots, r\}$. The u_k may be eliminated with the help of the r control (in)equations (f.2.18) leaving 2n equations in 2n unknowns. Note that the use of the maximum principle derived, does not involve the lengthy derivation for each design problem; the results applicable for handling the design problem are contained in the summary statement (§F.2.3).

The necessary conditions essentially constitute an aid in the search for the optimal control. The conditions are in general only <u>local</u> necessary conditions (see for example Athans 1966, Paiewonsky 1965 for discussion) and in this sense the derived controls are referred to as <u>extremal</u> controls in contrast to the wording <u>optimal</u> controls. The desired <u>global</u> (= optimal) control is the extremal control which gives the minimum value to the criterion. However, as extremal controls may not be unique so the optimal control may not be unique. Nonunique controls are only embarrassing computationally but not so in application. Certain controls are infeasible owing to the presence of engineering and other constraints that are not included in the mathematical statement of the design problem, and the possibility of several alternatives suggests in such a case that not only will a theoretical optimal solution be found but also one satisfying practical requirements.

The sufficiency of the conditions has not been proven. Such proofs are extremely difficult and at the present time are only available for certain restricted classes of systems and criteria. See for example Pontryagin et al (1962), Lee and Markus (1967) and Lee (1963). Proofs of sufficiency conditions for general nonlinear systems are very difficult. Similar statements apply in a discussion on existence. (Existence theorems imply extremal controls are optimal.) In preference to showing that the problem satisfies some existence theorem, an area with few useful results at the present time, the maximum principle has been derived assuming that an optimal control does indeed exist. (On the other hand, if a solution can be found, then that solution will be the desired one.) The principle is valid only under such assumptions. For a discussion on existence, see for example the last referenced works and in addition Filippov (1963) and Lee and Markus (1961).

The adjoint equations $(f.2.8)^2$ together with the system equations $(f.2.8)^1$ constitute the so-termed 'Hamilton canonical' differential equations when expressed in terms of the Hamiltonian (see Sage 1968). Coupling is through the control (in)equation (f.2.18). The terminology derives from an analogous construction of equations in analytical mechanics for lumped parameter systems - see §A.2 for the form of the canonical equations and their interpretation. (In the analogous mechanics equations, the Hamiltonian is an energy function, $\underline{x}$ are generalised coordinates, $\underline{\lambda}$ are generalised momenta and y is time; the state space is 2n-dimensional. See Rozonoer 1959.)

The maximum principle can be shown (for example Blum 1967, Berkovitz 1961, Kalman 1963b, A.I. Egorov 1966 among others) to lead to the three necessary conditionsof the calculus of variations - Euler-Lagrange, Legendre, and Weierstrass - as a direct consequence of the principle statement when no constraints on the control are present. The Weierstrass-Erdmann corner conditions, transversality conditions and natural boundary conditions follow similarly. A fourth necessary condition, the Jacobi condition, is not contained within the principle (Leitmann 1966). There is a correspondence between the adjoint variables (→ adjoint equations) of the maximum principle and the Lagrange multipliers (→ Euler-Lagrange equations) of the calculus of variations which is only applicable for unconstrained control (see §L.3 for a specific illustration, and §J).

Further extensions and complications of the necessary conditions are possible (for example the inclusion of state constraints, variable parameter intervals $[y_i^L, y_i^R]$ with reference to the time interval); the present work in no way exhausts all the possibilities but encompasses the results required for treating a large class of design problems. It is seen that superficially different design problems share a common mathematical basis and may be treated analogously. The theory is illustrated in the following section for a problem in plate design.

§G A DESIGN ILLUSTRATION: SYSTEM MODEL TYPE I

G.1 GENERAL

To illustrate and clarify the use of the necessary conditions for system model type I, a (distributed parameter) problem posed by Armand (1972) has been chosen. The problem is one of optimal control of the thickness distribution of a freely vibrating plate such as to minimize the mass of the structure with a bound on the fundamental frequency of vibration of the reference plate. A related exercise using a steepest descent algorithm has been given by Haug, Pan and Streeter (1972). The present solution uses the distributed parameter system extension of Pontryagin's necessary conditions for system type I (§F). For completeness sections §I and §K treat the same design problem when the plate is modelled as a system type II and III respectively. (The corresponding necessary conditions are derived in §H and §J.) Armand (1972) formulates the problem in the format of a particular type II system but does not give the solution.

It is noted that the reduction to a system type I format for this structure (assuming variable thickness) can only be accomplished in one way - namely according to appendix one where the mathematical meaning alone is satisfied. Choosing all variables, in the reduction, with physical meanings creates state equations with derivatives of the control appearing on the right hand side, the choice of control under such conditions in order that the Hamiltonian is maximized being unclear. However in section §I it is shown that several reductions are possible to obtain a type II format; the first after Armand where the variables lack physical meaning and which is analogous to the treatment in this section; the second as detailed in §I where physical significance is given to the choice of equations and variables involved (that is after §C.3); and another modifying this second reduction to a form after Lurie (1963) by eliminating state derivatives on the right hand side. In section §K a further three reductions are given; all three are of type III form and satisfy the mathematics alone, only differing in the order of the state derivatives on the right hand side and the number of state equations. It is remarked that all seven formulations illustrated (clearly further formulations are possible) lead ultimately to the same equations to be solved for optimality and further are singular to varying degrees. A following section (§L) elaborates on this last observation but for the moment the requirements for optimality in the presence of singularities will be shown to be implicitly satisfied by suitable substitutions made in the courses of the solutions.

The questions of physical significance and singularities aside, this section shows the manipulations required in order to design systems having been reduced to a type I form. A sensitivity analysis of the design results is given following the treatment of the problem in §I. There the deviation of the system behaviour and properties from the optimal caused by deviations in the system parameters is examined.

G.2 PROBLEM FORMULATION

G.2.1 The basic data. The constitutive relationship for a plate of variable thickness under free vibration reads

$$\frac{\partial^2}{\partial y_1^2}\left[D\frac{\partial^2 w}{\partial y_1^2} + D\nu\frac{\partial^2 w}{\partial y_2^2}\right] + \frac{\partial^2}{\partial y_2^2}\left[D\frac{\partial^2 w}{\partial y_2^2} + D\nu\frac{\partial^2 w}{\partial y_1^2}\right] + 2\frac{\partial^2}{\partial y_1 \partial y_2}\left[D(1-\nu)\frac{\partial^2 w}{\partial y_1 \partial y_2}\right] + \rho h\frac{\partial^2 w}{\partial t^2} = 0 \tag{g.2.1}$$

Notation, generally, and conventions follow the work of Timoshenko and Woinowsky-Krieger (1959) with the inertia term appended in accordance with d'Alemberts principle. In particular,

- D denotes the flexural rigidity, $Eh^3/12(1-\nu^2)$
- h the plate thickness, $h(y_1, y_2)$
- E the modulus of elasticity
- ν the Poisson's ratio
- w the lateral displacement, $w(y_1, y_2)$, from the equilibrium position
- ρ the material density (mass per unit area), $\rho(y_1, y_2)$
- y_1, y_2 position coordinates, $0 \leq y_1 \leq a$, $0 \leq y_2 \leq b$
- t time

The classical solution of the eigenvalue problem assumes a variables separable form

$$w(y_1, y_2, t) = W(y_1, y_2)f(t) \tag{g.2.2}$$

where the variables W and f are functions of the given arguments. f is a time dependent harmonic function, $\cos(\omega t)$, where ω is the circular frequency (expressed in radians per unit time).

Substitution in equation (g.2.1) yields

$$\text{(g.2.3)} \qquad \frac{\partial^2}{\partial y_1^2}\left[D\,\frac{\partial^2 W}{\partial y_1^2} + D\nu\,\frac{\partial^2 W}{\partial y_2^2}\right] + \frac{\partial^2}{\partial y_2^2}\left[D\,\frac{\partial^2 W}{\partial y_2^2} + D\nu\,\frac{\partial^2 W}{\partial y_1^2}\right]$$

$$+\ 2\frac{\partial^2}{\partial y_1 \partial y_2}\left[D(1-\nu)\,\frac{\partial^2 W}{\partial y_1 \partial y_2}\right] - e^2 D^{\frac{1}{3}} W = 0$$

where $e^2 = \rho\omega^2\,[12(1-\nu^2)/E]^{\frac{1}{3}} \geq 0$

In general, for any given plate geometry, the solution yields an infinite sequence of eigenvalues from which the smallest or fundamental frequency, ω_f, may be obtained. For the problem at hand, a uniform reference plate is chosen. The variable thickness plate is then constrained to satisfy equation (g.2.3) with the inertia term appropriately modified. Boundary and additional constraints remain unchanged.

A minimum mass solution is sought, implying an optimality criterion

$$\text{(g.2.4)} \qquad Q = \int_{y_2}\int_{y_1} \rho h \, dy_1 dy_2$$

To exploit the notational simplicity of retaining the term D (flexural rigidity) in the system equation, (equation (g.2.3) as modified), in preference to the more usual design parameter h (plate thickness), the criterion is converted to the form

$$\text{(g.2.5)} \qquad Q = \int_{y_2}\int_{y_1} D^{\frac{1}{3}} dy_1 dy_2$$

where certain constant multiplicative terms have been omitted without loss of accuracy. Similarly no rigour is lost in the notational simplification.

Specializing the above; for a uniform reference rectangular plate, freely supported, the natural frequencies are given by (see for example Leissa 1969)

$$\omega_{mn} = \sqrt{\frac{D_o}{\rho h_o}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \tag{g.2.6}$$

where m and n are integers and the subscript o denotes the constant form of the variable corresponding to the reference plate properties. Isometry and homogeneity properties have been invoked in all plate equations.

From equation (g.2.6) the fundamental frequency is found to be

$$\omega_f = \omega_{11} = \sqrt{\frac{D_o}{\rho h_o}}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right] \tag{g.2.7}$$

Boundary conditions imply

$$\begin{aligned} W\Big|_{y_1 = 0,a} &= 0, \quad D\frac{\partial^2 W}{\partial y_1^2} + D\nu\frac{\partial^2 W}{\partial y_2^2}\Big|_{y_1 = 0,a} = 0 \\ W\Big|_{y_2 = 0,b} &= 0, \quad D\frac{\partial^2 W}{\partial y_2^2} + D\nu\frac{\partial^2 W}{\partial y_1^2}\Big|_{y_2 = 0,b} = 0 \end{aligned} \tag{g.2.8}$$

where the coordinate axes have been chosen to coincide with the plate sides.

The optimal problem now assumes the form; minimize the criterion (equation (g.2.5)), subject to the constraints (equation (g.2.3) as modified by equation (g.2.7); equation $(g.2.8)^1$, and equation $(g.2.8)^2$).

<u>G.2.2 State-control interpretation.</u> Introduce a state vector $\underline{x}$ and control u according to the algorithm in appendix one,

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \triangleq \begin{bmatrix} W \\ \dfrac{\partial W}{\partial y_1} \\ \dfrac{\partial^2 W}{\partial y_1^2} \\ \dfrac{\partial^3 W}{\partial y_1^3} \\ D \\ \dfrac{\partial D}{\partial y_1} \end{bmatrix}$$

$$u \triangleq \frac{\partial^2 D}{\partial y_1^2}$$

such that by differentiation with respect to y_1

$$\begin{bmatrix} \dfrac{\partial x_1}{\partial y_1} \\ \dfrac{\partial x_2}{\partial y_1} \\ \dfrac{\partial x_3}{\partial y_1} \\ \dfrac{\partial x_4}{\partial y_1} \\ \dfrac{\partial x_5}{\partial y_1} \\ \dfrac{\partial x_6}{\partial y_1} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ -\dfrac{\theta}{x_5} \\ x_6 \\ u \end{bmatrix} \tag{g.2.9}$$

where $$\theta = ux_3 + 2x_6x_4 + \nu\left(u\frac{\partial^2 x_1}{\partial y_2^2} + 2x_6\frac{\partial^2 x_2}{\partial y_2^2} + x_5\frac{\partial^2 x_3}{\partial y_2^2}\right)$$

$$+ 2(1-\nu)\frac{\partial}{\partial y_2}\left(x_6\frac{\partial x_2}{\partial y_2} + x_5\frac{\partial x_3}{\partial y_2}\right) + \frac{\partial^2}{\partial y_2^2}\left(x_5\frac{\partial^2 x_1}{\partial y_2^2} + \nu x_5 x_3\right)$$

$$- e^2 x_5^{\frac{1}{3}} x_1$$

Notice that the choice of independent variable involved in the derivative terms of the state vector (and hence the differentiation on the left hand side of (g.2.9)) is arbitrary as the problem is symmetrical with respect to both y_1 and y_2.

State boundary conditions are:

(g.2.9a)
$$x_1\Big|_{y_1 = 0,a} = 0 \qquad x_5x_3 + \nu x_5\frac{\partial^2 x_1}{\partial y_2^2}\Big|_{y_1 = 0,a} = 0$$

$$x_1\Big|_{y_2 = 0,b} = 0 \qquad x_5\frac{\partial^2 x_1}{\partial y_2^2} + \nu x_5 x_3\Big|_{y_2 = 0,b} = 0$$

The criterion similarly becomes

(g.2.10) $$Q = \int_0^b \int_0^a x_5^{\frac{1}{3}} dy_1 dy_2$$

With the system differential equations (g.2.9) and the criterion (g.2.10) now in hand, the Hamiltonian may be written

(g.2.11) $$H = - x_5^{\frac{1}{3}} + \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3 x_4 - \frac{\lambda_4\theta}{x_5} + \lambda_5 x_6 + \lambda_6 u$$

The maximum principle requires that u must be chosen for all (y_1, y_2) such that H(u) is maximized. However it is noted that the control appears linearly in the Hamiltonian with coefficient

(g.2.12) $$\sigma(y_1,\ y_2) = \frac{-\lambda_4}{x_5}\left\{x_3 + \nu\frac{\partial^2 x_1}{\partial y_2^2}\right\} + \lambda_6$$

This is equivalent to a singular condition for the maximum principle formulation (see §L for a complete treatment of singularities) and without control constraints, it is required that σ be maintained at zero over the optimal solution. A formal solution along these lines however will not be undertaken but instead a simplifying substitution will be later used that implicitly satisfies the requirements for optimality in the presence of singularities. (For a formal approach to the question of singularities, see §L.) Optimality of the resulting solution is shown later in §L and also in §I by another approach.

The adjoint equations read

$$
\text{(g.2.13)} \qquad
\begin{bmatrix}
\dfrac{\partial\lambda_1}{\partial y_1} \\[3ex]
\dfrac{\partial\lambda_2}{\partial y_1} \\[3ex]
\dfrac{\partial\lambda_3}{\partial y_1} \\[6ex]
\\[3ex]
\dfrac{\partial\lambda_4}{\partial y_1} \\[3ex]
\dfrac{\partial\lambda_5}{\partial y_1} \\[6ex]
\\[3ex]
\dfrac{\partial\lambda_6}{\partial y_1}
\end{bmatrix}
=
\begin{bmatrix}
\nu \dfrac{\partial^2}{\partial y_2^2}(u\phi) + \dfrac{\partial^2}{\partial y_2^2}\left(x_5 \dfrac{\partial^2\phi}{\partial y_2^2}\right) - e^2 x_5^{\frac{1}{3}}\phi \\[3ex]
-\lambda_1 + 2\nu \dfrac{\partial^2}{\partial y_2^2}(x_6\phi) + 2(1-\nu)\dfrac{\partial}{\partial y_2}\left(x_6 \dfrac{\partial\phi}{\partial y_2}\right) \\[3ex]
-\lambda_2 + u\phi + \nu \dfrac{\partial^2}{\partial y_2^2}(x_5\phi) + 2(1-\nu)\dfrac{\partial}{\partial y_2}\left(x_5 \dfrac{\partial\phi}{\partial y_2}\right) \\[3ex]
+ \nu x_5 \dfrac{\partial^2\phi}{\partial y_2^2} \\[3ex]
-\lambda_3 + 2x_6\phi \\[3ex]
\dfrac{1}{3} x_5^{-\frac{2}{3}} - \dfrac{\phi\theta}{x_5} + \nu\phi \dfrac{\partial^2 x_3}{\partial y_2^2} - 2(1-\nu)\dfrac{\partial\phi}{\partial y_2}\dfrac{\partial x_3}{\partial y_2} \\[3ex]
+ \dfrac{\partial^2\phi}{\partial y_2^2}\dfrac{\partial^2 x_1}{\partial y_2^2} + \nu x_3 \dfrac{\partial^2\phi}{\partial y_2^2} - \dfrac{1}{3} e^2 x_5^{-\frac{2}{3}} x_1 \phi \\[3ex]
-\lambda_5 + 2x_4\phi + 2\phi \dfrac{\partial^2 x_2}{\partial y_2^2} - 2(1-\nu)\dfrac{\partial\phi}{\partial y_2}\dfrac{\partial x_2}{\partial y_2}
\end{bmatrix}
$$

where $\phi = \lambda_4/x_5$

with natural boundary and transversality conditions

$$\lambda_2\Big|_{y_1 = 0,a} = 0, \qquad \lambda_4\Big|_{y_1 = 0,a} = 0, \qquad \lambda_6\Big|_{y_1 = 0,a} = 0,$$

$$\lambda_3 + \Lambda_{0,a}\, x_5\Big|_{y_1 = 0,a} = 0, \qquad \lambda_5 + \Lambda_{0,a}\left[x_3 + \nu\,\frac{\partial^2 x_1}{\partial y_2^2}\right]\Bigg|_{y_1 = 0,a} = 0,$$

(where $\Lambda_{0,a}$ are Lagrange multipliers (§F.3))

(g.2.13a) $$\left[2\nu\,\frac{\partial}{\partial y_2}(x_6\phi) + 2(1-\nu)\,x_6\,\frac{\partial \phi}{\partial y_2}\right]\Big[\delta x_2\Big]_{y_2 = 0,b} = 0,$$

$$\left[2\,\frac{\partial \phi}{\partial y_2}\,x_5\right]\Big[\delta x_3\Big]_{y_2 = 0,b} = 0,$$

$$\left[-\,2\phi\,\frac{\partial}{\partial y_2}\left(x_3 + \frac{\partial^2 x_1}{\partial y_2^2}\right) + \frac{\partial}{\partial y_2}\left(\phi\,\frac{\partial^2 x_1}{\partial y_2^2} + \nu\phi x_3\right)\right]\Big[\delta x_5\Big]_{y_2 = 0,b} = 0,$$

$$\left[-\,2(1-\nu)\,\phi\,\frac{\partial x_2}{\partial y_2}\right]\Big[\delta x_6\Big]_{y_2 = 0,b} = 0$$

G.3 SOLUTION OF THE NECESSARY CONDITIONS

Equations (g.2.9) (including conditions (g.2.9a)) and (g.2.13) (including (g.2.13a)) constitute the necessary relations that have to be solved for an optimal solution to be gained of the problem. There are twelve equations relating thirteen unknowns $x_i(y_1, y_2)$, $\lambda_i(y_1, y_2)$, $i=1, \ldots, 6$ and $u(y_1, y_2)$. The equations are linked by the requirement that the Hamiltonian be maximized with respect to the control. Generally this will give the additional equation (relating the optimal control to the state and adjoint variables) needed to determine the solution completely. However as the control appears linearly in H, this singular condition requires that the coefficient of u in H be maintained at zero for optimality (see §L).

By inspection of the construction of equations (g.2.9) and (g.2.13) it is seen that

$$
\text{(g.3.1)} \qquad
\begin{bmatrix} \lambda_1 \\ \\ \lambda_2 \\ \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix}
= \alpha^2
\begin{bmatrix}
\dfrac{\partial}{\partial y_1}\left[x_5 x_3 + \nu x_5 \dfrac{\partial^2 x_1}{\partial y_2^2}\right] + 2(1-\nu)\dfrac{\partial}{\partial y_2}\left[x_5 \dfrac{\partial x_2}{\partial y_2}\right] \\
+ \nu \dfrac{\partial^2}{\partial y_2^2}\left[x_5 x_2 - x_1 x_6\right] \\
-\left[x_5 x_3 + \nu x_5 \dfrac{\partial^2 x_1}{\partial y_2^2}\right] - 2(1-\nu)\dfrac{\partial}{\partial y_2}\left[x_5 \dfrac{\partial x_1}{\partial y_2}\right] \\
- \nu \dfrac{\partial^2}{\partial y_2^2}\left[x_5 x_1\right] \\
x_5 x_2 - x_1 x_6 \\
-x_5 x_1 \\
\left[x_2 x_3 - x_1 x_4\right] + \nu\left[x_2 \dfrac{\partial^2 x_1}{\partial y_2^2} - x_1 \dfrac{\partial^2 x_2}{\partial y_2^2}\right] + 2(1-\nu)\dfrac{\partial x_1}{\partial y_2}\dfrac{\partial x_2}{\partial y_2} \\
- x_1 x_3 - \nu x_1 \dfrac{\partial^2 x_1}{\partial y_2^2}
\end{bmatrix}
$$

is a valid substitution, where α^2 is an undetermined constant and may be thought of as an amplitude factor on the states. (Notice that the system equation (g.2.3) is homogeneous in W. The states related to W and its derivatives are thus only determined to within a constant.) The choice of a constant squared will become apparent on comparison with the solution given in section §I. These relations (g.3.1) are also compatible with the boundary conditions on $\underline{\lambda}$ and implicitly maintain σ zero over the domain of the plate. (Uniqueness of the solution is assumed and hence is the desired optimal solution.) By relating the state and adjoint variables in this way, the manipulations normally required in the singular case have been circumvented. The problem has also been reduced in size.

With these substitutions, $(g.2.13)^1$ is equivalent to the original system equation (g.2.3), $(g.2.13)^{2,3,4,6}$ are identities, and $(g.2.13)^5$ becomes

$$\frac{\partial^2 W}{\partial y_1^2}\left(\frac{\partial^2 W}{\partial y_1^2} + \nu\,\frac{\partial^2 W}{\partial y_2^2}\right) + 2(1-\nu)\left(\frac{\partial^2 W}{\partial y_1 \partial y_2}\right)^2 + \frac{\partial^2 W}{\partial y_2^2}\left(\frac{\partial^2 W}{\partial y_2^2} + \nu\,\frac{\partial^2 W}{\partial y_1^2}\right)$$

(g.3.2)

$$- \frac{1}{3\alpha^2}\,[1 + \alpha^2 e^2 W^2]\,D^{-\frac{2}{3}} = 0$$

using (g.2.3) after substituting for the original variables W and D. That is the optimal solution is found by the simultaneous solution of (g.2.3) (or (g.2.9)) and (g.3.2). Boundary conditions (g.2.8) apply.

Sundry equation solving techniques may be used; an approximate numerical procedure, with discretization over one of the spatial variables (y_1) is employed here. There remains to be solved, a finite dimensional system or ordinary differential equations continuous over the other spatial variable (y_2).

Using the central difference expression (§E.2)

$$\frac{dx_i}{dy_1} = (x_i^{k+1} - x_i^k)/\Delta$$

in the y_1 direction at k, equations (g.2.9) and (g.3.2) become respectively

(g.3.3)
$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \\ x_3^{k+1} \\ x_4^{k+1} \\ x_5^{k+1} \\ x_6^{k+1} \end{bmatrix} = \begin{bmatrix} x_1^k + \Delta x_2^k \\ x_2^k + \Delta x_3^k \\ x_3^k + \Delta x_4^k \\ x_4^k - \Delta\,\dfrac{\theta^k}{x_5^k} \\ x_5^k + \Delta x_6^k \\ x_6^k + \Delta u^k \end{bmatrix}$$

where
$$\theta^k = u^k x_3^k + 2x_6^k x_4^k + \nu\left(u^k \frac{d^2x_1^k}{dy_2^2} + 2x_6^k \frac{d^2x_2^k}{dy_2^2} + x_5^k \frac{d^2x_3^k}{dy_2^2}\right)$$
$$+ 2(1-\nu)\frac{d}{dy_2}\left(x_6^k \frac{dx_2^k}{dy_2} + x_5^k \frac{dx_3^k}{dy_2}\right) + \frac{d^2}{dy_2^2}\left(x_5^k \frac{d^2x_1^k}{dy_2^2} + \nu x_5^k x_3^k\right)$$
$$- e^2 (x_5^k)^{\frac{1}{3}} x_1^k \; .$$

(g.3.4)
$$x_4\left(x_3 + \nu\frac{d^2x_1}{dy_2^2}\right) + 2(1-\nu)\left(\frac{dx_2}{dy_2}\right)^2 + \frac{d^2x_1}{dy_2^2}\left(\frac{d^2x_1}{dy_2^2} + \nu x_3\right)$$
$$- \frac{1}{3\alpha^2}\,[1 + \alpha^2 e^2 x_1^2]\, x_5^{-\frac{2}{3}} = 0$$

where the discrete form of the variables now only have arguments of y_2.

Equations (g.3.3) and (g.3.4) are solved sequentially for each k and represent transitions between functions of state $\underline{x}(y_2)$ at k and k + 1.

The problem as defined is of the 'boundary value' type implying that the boundary conditions are specified at more than one value of the independent variables y_1 and y_2. The adopted solution procedure however, was to estimate certain unknown boundary conditions, so as to convert the problem into an 'initial-value' type and then to iterate on the known boundary conditions specified elsewhere to the computations starting point. The initial estimates of the unknown boundary conditions at the starting point were modified until all given boundary conditions were satisfied.

To obtain definite results, a nominal e^2 value of 1.0 (corresponding to an aluminium reference plate 10" x 10" x 50/1000" - see Armand 1972) has been chosen in the computations. For this particular case a good approximate numerical solution for W and D is given in table G.3.1 for $\alpha^2 = 1$.

y_2 \ y_1	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5
5.0	1.094									
	169.2									
5.5	1.094	1.094								
	169.2	169.2								
6.0	1.071	1.071	1.049							
	167.0	167.0	164.6							
6.5	1.023	1.023	1.001	0.957						
	160.4	160.4	158.1	152.4						
7.0	0.952	0.952	0.932	0.890	0.828					
	149.3	149.3	147.1	141.8	131.9					
7.5	0.856	0.856	0.838	0.801	0.745	0.670				
	133.4	133.4	131.4	126.7	117.8	105.3				
8.0	0.738	0.738	0.722	0.690	0.642	0.577	0.497			
	112.8	112.8	111.2	107.2	99.7	89.0	75.3			
8.5	0.593	0.593	0.581	0.555	0.516	0.464	0.400	0.322		
	88.0	88.0	86.7	83.6	77.7	69.5	58.8	45.8		
9.0	0.425	0.425	0.416	0.398	0.371	0.333	0.287	0.230	0.166	
	60.1	60.1	59.3	57.1	53.1	47.5	40.1	31.3	21.4	
9.5	0.230	0.230	0.225	0.215	0.200	0.180	0.155	0.124	0.090	0.048
	31.9	31.9	31.5	30.3	28.2	25.2	21.3	16.6	11.4	6.0

Table G.3.1: Calculated values of displacement W (upper numbers, inches) and rigidity D (lower numbers, lb inches) for illustration problem. y_1 and y_2 values are in inches.

G.4 COMMENT

The solution technique, namely the distributed parameter extension of Pontryagin's maximum principle, provides a quite elegant and useful means of designing deterministic systems (in the present case, systems of type I form). It is systematic and reduces, when used in its most general sense, to purely mechanical handling of several fundamental relations.

The scope of application of the technique is not limited to problems of the type illustrated here; rather the essential method of solution remains the same for a diversity of problems. The application potential is only limited by a certain awkwardness and involvedness in obtaining answers for complicated problems; it appears that some form of numerical solution procedure would have to be invoked for all but the most elementary problems.

Determining the optimal design involved solving a set of necessary conditions. These conditions were simultaneous partial differential equations of the boundary value type. These equations give a general prescription of the solution, from which a particular numerical solution could be obtained.

§H DERIVATION OF NECESSARY CONDITIONS FOR OPTIMALITY: SYSTEM TYPE II

H.1 INTRODUCTION

H.1.1 Outline. This section derives the conditions for the optimal control of the second form of structural models considered in this thesis, namely systems type II. The concept of dynamic programming (Bellman 1957a, 1961, Bellman and Dreyfus 1962) is used to derive a partial differential functional equation expressing the conditions of optimality. This equation is derived from a recurrence relation using the principle of optimality and an imbedding procedure (whereby the original problem is replaced by a sequence of smaller problems). The parameters defining the smaller problems are the state defined on a one-parameter family of surfaces and the parameter of this family of surfaces. The variation is taken in the surfaces' parameter. The resulting equation expressing optimality is a distributed parameter system generalisation of Bellman's equation for lumped systems (equivalently the Hamilton-Jacobi partial differential equation of variational calculus). The Hamilton-Jacobi-Bellman equation is both necessary and sufficient.

The Hamilton-Jacobi-Bellman equation is reinterpreted in the form of a distributed parameter system generalisation of Pontryagin's maximum principle (analogous to the results of §F and §J) and the Hamilton canonical equations. Here the necessary conditions for optimality appear (as in §F and §J) as a set of auxiliary equations simultaneous to the system equations. Boundary conditions for these auxiliary equations appear as additional conditions to be satisfied. The equations in this form are of a boundary value type. (Compare with the Hamilton-Jacobi-Bellman equation which is an initial value type.)

As in §F, a basic problem is studied, necessary conditions obtained, and then certain extensions incorporated. The ultimate form of the results is applicable to a broad class of design problems. Bellman's functional equation is applicable for problems with constraints and without constraints yielding a globally optimum solution. On transferring to a maximum principle formulation, a distinction between these types of problems has to be made and it is found most convenient to treat the unconstrained problem first and then generalize to the constrained case.

The conditions for type II systems presented here, combined with the conditions of §F for type I and §J for type III, will cover a very broad class of problems likely to be encountered in structural design. The comments made in §F and §J regarding the approach to design are valid here.

Through deriving a version of the maximum principle from the equations of dynamic programming, the essentially interchangeable nature of the three approaches (in §F, §H and §J) is well illustrated. By adopting different routes to the derivation of the maximum principle, the designer will be made more aware of his assumptions and limitations of the results. It is also intuitively satisfying to show some form of unification between the techniques. For a more detailed discussion of the connection between variational approaches and dynamic programming, see Dreyfus (1965), Bellman (1961), and Lee (1964) in particular.

H.1.2 Background. In discussing the backgrounds to the derivations of the other deterministic conditions for optimality (§F for systems type I, §J for systems type III), it is noted that no results are available in the literature dealing with the cases required for the structural design problem posed. The same comment may be reiterated here. The derivations of Lurie (1963) (see also Butkovskii et al 1968, Armand 1971, 1972) although applied to systems of a similar form were found to be inadequate for the writer's system model type II in that they were only two dimensional in nature and did not allow derivatives of state on the right hand sides. (These authors have avoided having derivatives on the right hand sides by introducing additional dependent variables to replace the derivative terms. See §C and the following illustration in §I for further discussion.) The case treated by these authors is the multiple (double) integral case of the calculus of variations and leads to a two dimensional generalization of the Euler-Lagrange equations and other necessary conditions. (See Gelfand and Fomin 1963 for example.) (The Euler-Lagrange equations for the m-dimensional case are sometimes referred to as the Ostrogradski equations - see Elgolc 1961.)

The utilization of dynamic programming concepts in the study of the optimality of distributed parameter systems was initially suggested by Bellman in association with Osborn (1958) and Kalaba (1961, 1962). These fundamental studies were subsequently advanced upon by Wang and Tung (1964) and Wang (1964) who derive necessary conditions for a general system described by partial differential equations similar to system model type I. (See also Brogan 1967a,b, 1968a, 1968b for the extension of Wang's results to include boundary controls.)

Generally, the above authors increment their problem over time only. Angel and Bellman (1972) (see also Angel 1968a, 1968b, 1970) suggest minimization problems over regions be formulated in a dynamic programming sense through the device of minimizing over subregions. (They were specifically concerned with the Dirichlet functional resulting from the potential equation. Distefano (1971) on the biharmonic equation uses a related device.) The choice of the subregion dictates the form of the final results. If an infinitesimal is chosen, as in the present section, a differential equation results.

For particular applications of dynamic programming to control problems see Erzberger and Kim (1966a, 1966b), Kim and Erzberger (1967) and Butkovskii (1969). All are related to linear systems and are reduced to a Hamilton-Jacobi-Bellman equation to solve for optimality.

In the following article (§H.2), the format of the derivation will closely follow the lumped parameter treatment of Long (1972).

H.2 DERIVATION OF THE OPTIMALITY CONDITIONS

H.2.1 Preliminary notes and assumptions. Consider a system described over a closed region Y in the $y_1y_2y_3$ - space with piecewise smooth boundary surfaces ∂Y^a and ∂Y^b (figure H.2.1, one octant shown only).

The system equations will be taken to be of the form (§C.3)

$$\text{(h.2.1)} \qquad \frac{\partial \underline{x}}{\partial y_i} = \underline{f}^i[\underline{y}, \underline{x}, \ldots, \partial_{\underline{\ell}}\underline{x}, \ldots, \underline{u}] \qquad i = 1, 2, 3$$

where $\underline{\ell} = (\ell_h, \ell_k)$ $(h, k = 1, 2, 3;\ h, k \neq i)$ and $\partial_{\underline{\ell}}\underline{x}$ is as defined in the 'notation'. $\underline{x}(\underline{y}) = (x_1, \ldots, x_n)^T$ denotes the state and $\underline{u}(\underline{y}) = (u_1, \ldots, u_r)$ the control at any $\underline{y} \in Y$, $\underline{y} = (y_1, y_2, y_3)^T$. $\underline{f}^i$, $i = 1, 2, 3$, are in general nonlinear vector-valued functions of the arguments shown and have to be such that they satisfy certain compatibility conditions (§C.3).

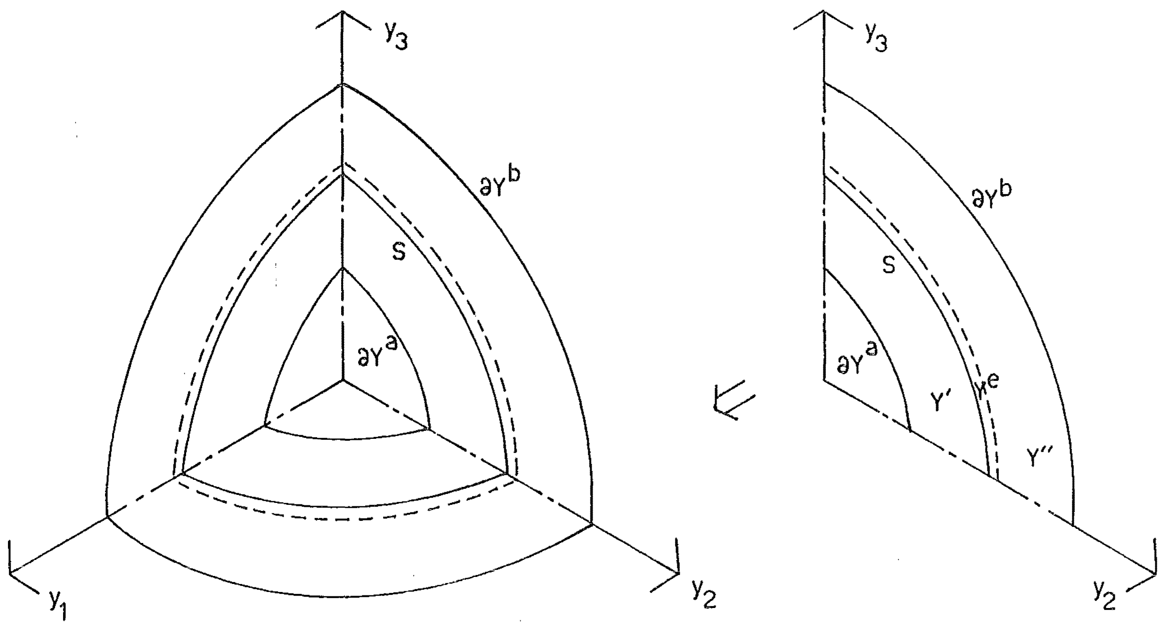

Figure H.2.1

The set of available controls will not usually be arbitrary but will be constrained to some admissible region U (§D.2). A control $\underline{u}(\underline{y})$ will be said to be admissible if

(h.2.2) $\qquad \underline{u}(\underline{y}) \in U(\underline{u}) \qquad\qquad \forall \underline{y} \in Y$

and $\underline{u}(\underline{y})$ is piecewise continuous in Y.

State boundary conditions will be specified on ∂Y^a and ∂Y^b of the form

(h.2.1a) $\qquad \partial_{\underline{\ell}-1} x_j$ given

and bear a direct relationship to the $\partial_{\underline{\ell}}\underline{x}$ terms appearing on the right hand side of (h.2.1). (This follows from the choice of state variables.) The values taken by j in (h.2.1a) are determined by the conditions of any given problem.

It is assumed that the state may be found uniquely from (h.2.1) and (h.2.1a) for any given admissible control. Notice the system equations (h.2.1) are more general than in Lurie (1963) (or Armand 1971, 1972) in that the equations are described over an extra dimension and they also permit derivatives of state on the right hand sides.

Alternative controls will be taken to be evaluated according to the optimality criterion (§D.3)

(h.2.3) $$Q = \int_Y G(\underline{y}, \underline{x}, \ldots, \partial_{\underline{\ell}}\underline{x}, \ldots, \underline{u})d\underline{y}$$

The optimal (admissible) control is so chosen as to minimize the functional Q.

The functional equation approach of dynamic programming imbeds this minimization problem within a family of problems with 'initial' states and locations of these initial states over Y as parameters.

<u>H.2.2 Derivation of the functional equation.</u> Consider the region Y divided into two subregions Y' and Y" separated by a closed surface S(figure H.2.1) belonging to a one parameter family of surfaces

(h.2.4) $$\Phi(y_1, y_2, y_3, c) = o$$

where c is the parameter of the family. S can be reduced to the boundary surfaces ∂Y^a and ∂Y^b by a continuous deformation. Appendix three shows that such a family can be constructed using for example a spherical polar coordinate system. On S, the areal measurement $s = s(y_1, y_2, y_3)$ and the parameter $c = c(y_1, y_2, y_3)$ which can be solved for y_1, y_2 and y_3 to yield $y_1 = y_1(s,c)$, $y_2 = y_2(s,c)$ and $y_3 = y_3(s,c)$; that is $\underline{x}(\underline{y}) \to \underline{x}(s, c)$, $\underline{u}(\underline{y}) \to \underline{u}(s, c)$ on S.

Define

(h.2.3a) $$Q^*[\underline{x},\underline{u},c] = \int_{Y''} G(\underline{\gamma}, \underline{x}(\underline{\gamma}), \ldots, \partial_{\underline{\ell}}\underline{x}(\underline{\gamma}), \ldots, \underline{u}(\underline{\gamma}))d\underline{\gamma}$$

That is Q^* is the criterion evaluated over the region Y" from the state $\underline{x}$ at S to the state at ∂Y^b determined by the (admissible) control

$\{\underline{u}(\gamma);\ \underline{\gamma} \in Y''\}$. Here $\underline{u}(\underline{\gamma})$ is arbitrary and independent of $\underline{x}(\underline{\gamma})$. Suppose now the optimal control $\hat{\underline{u}}$ is used. At each state $\underline{x}$, $\hat{\underline{u}}$ is determined by $\underline{x}$ and so $\hat{\underline{u}} = \hat{\underline{u}}(\underline{x})$. Then

$$\hat{Q}[\underline{x},c] = Q^*[\underline{x},\hat{\underline{u}},c] = \min_{\underline{u} \in U} Q^*[\underline{x},\underline{u},c] \qquad \text{(h.2.3b)}$$

The arguments of $\hat{Q}$, namely $\underline{x}$ and c, in this sense may be regarded as parameters defining a family of problems.

The integral defining Q^* may be expressed as the sum of two terms corresponding to an incremental portion over the region Y^e between two nearby members of the family of surfaces given by

$$\Phi(y_1,\ y_2,\ y_3,\ c) = o \quad , \quad \Phi(y_1,\ y_2,\ y_3,\ c+\delta c) = o$$

and the residual portion $Y^r = Y''-Y^e$

$$\hat{Q}[x,c] = \min_{\underline{u} \in U} \left\{ \int_{Y^e} G(\underline{y},\ \underline{x},\ \ldots,\ \partial_{\underline{\ell}}\underline{x},\ \ldots,\ \underline{u})d\underline{y} + \int_{Y^r} G(\underline{\nu},\ \underline{x},\ \ldots,\ \partial_{\underline{\ell}}\underline{x},\ \ldots,\ \underline{u})d\underline{\nu} + o(\delta c) \right\} \qquad \text{(h.2.5)}$$

Notice that the first term in the braces depends only on $\underline{u}(y)$ at the position $\underline{y}$, while the second term depends on $u(\underline{\nu})$ over the residual portion, $\underline{\nu} \in Y^r$.

A change of variables is made in the variables of integration of the first term from $(y_1,\ y_2,\ y_3)$ to (s,c) so that (h.2.5) becomes

$$\hat{Q}[\underline{x},c] = \min_{\underline{u} \in U} \left\{ \int_c^{c+\delta c} \int_S G[s,c,\underline{x}(s,c),\ \ldots,\ \partial_{\underline{\ell}}\underline{x}(s,c),\ \ldots,\ \underline{u}(s,c)]\,|J(s,c)|\,ds\,dc + \int_{Y^r} G[\underline{\nu},\underline{x}(\underline{\nu}),\ \ldots,\ \partial_{\underline{\ell}}\underline{x}(\nu),\ \ldots,\ \underline{u}(\nu)]\,d\underline{\nu} + o(\delta c) \right\} \qquad \text{(h.2.5b)}$$

where $|J(s,c)|$ denotes the Jacobian.

To evaluate this Jacobian when Φ is not defined explicitly consider figure H.2.2.

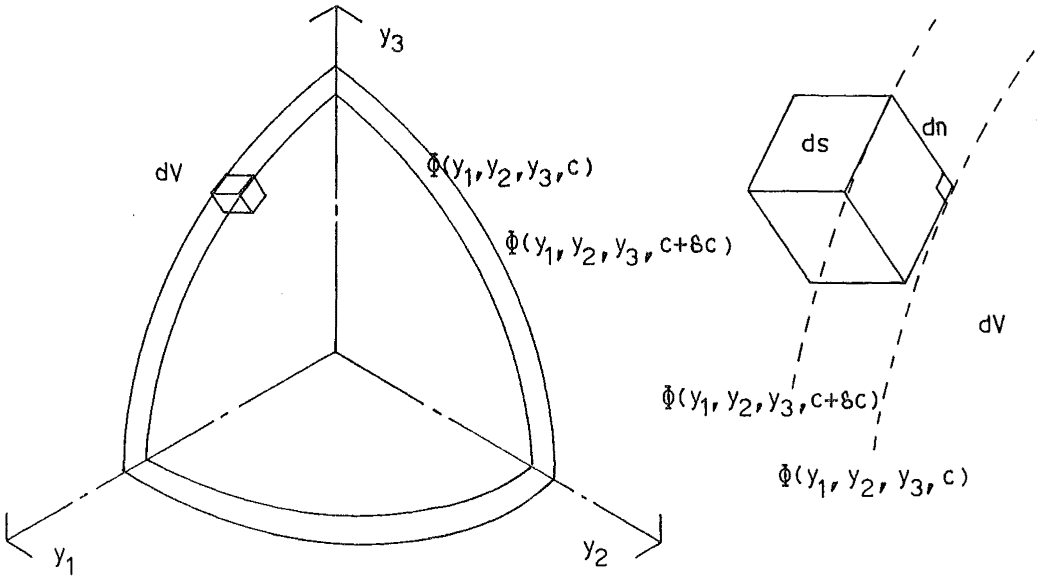

Figure H.2.2

Between $\Phi(y_1, y_2, y_3, c)$ and $\Phi(y_1, y_2, y_3, c+\delta c)$ the differentials dy_1, dy_2, dy_3 and dc are not all independent but are related by

$$\frac{\partial\Phi}{\partial y_1} dy_1 + \frac{\partial\Phi}{\partial y_2} dy_2 + \frac{\partial\Phi}{\partial y_3} dy_2 + \frac{\partial\Phi}{\partial c} dc = o$$

For an outward surface normal n

$$|\nabla\Phi|\,|dn| + \frac{\partial\Phi}{\partial c} dc = o$$

$$|dn| = \frac{\frac{-\partial\Phi}{\partial c} dc}{|\nabla\Phi|}$$

To preserve equal volumes under the transformation, to first order

$$dV = dy_1 dy_2 dy_3 = |dn|ds$$

$$= \frac{\left|\frac{\partial\Phi}{\partial c}\right| dcds}{|\nabla\Phi|}$$

$$= \frac{\left|\frac{\partial\Phi}{\partial c}\right|}{\left[\left(\frac{\partial\Phi}{\partial y_1}\right)^2 + \left(\frac{\partial\Phi}{\partial y_2}\right)^2 + \left(\frac{\partial\Phi}{\partial y_3}\right)^2\right]} \text{dcds}$$

$$= |J(s,c)| \text{ dcds}$$

For small δc and omitting terms $o(\delta c)$ of small order higher than δc

$$\hat{Q}[\underline{x},c] = \min_{\underline{u} \in U} \left\{ \delta c \int_S G(s,c,\underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}) |J(s,c)| ds \right.$$

$$\left. + \min_{\underline{u}(\underline{\nu}) \in U(\underline{u})} \int_{Y^r} G(\underline{\nu}, \underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}) d\underline{\nu} \right\}$$

(h.2.6)

$$= \min_{\underline{u} \in U} \left\{ \delta c \int_S G(s,c,\underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}) |J(s,c)| ds \right.$$

$$\left. + \hat{Q}[\underline{x}',c'] \right\}$$

where $c' = c+\delta c$

$x' = \underline{x}(s,c') = \underline{x}(s,c+\delta c)$

$\hat{Q}[\underline{x}',c'] = \hat{Q}[\underline{x}(s,c+\delta c), c+\delta c]$

The same result (equation h.2.6) could have been obtained more directly by applying Bellman's principle of optimality to (h.2.3) and (h.2.5).

The assumption is now made that $\hat{Q}$ has partial derivatives with respect to the states x_j and parameter c, and that the derivatives exist. The validity of this assumption may only be tested for individual cases, and in certain cases it may not be true.

$\hat{Q}[\underline{x}',c']$ may be expanded in the neighbourhood of $\hat{Q}[\underline{x},c]$ as follows (Wang 1964, Brogan 1968b),

$$\hat{Q}[\underline{x}',c'] = \hat{Q}[\underline{x},c] + \delta c \int_S \sum_{j=1}^{n} \frac{\partial \hat{Q}[\underline{x},c]}{\partial x_j} \frac{\partial x_j}{\partial c} ds + \frac{\partial \hat{Q}[\underline{x},c]}{\partial c} \delta c + o(\delta c) \tag{h.2.7}$$

Although the notation does not distinguish, $\dfrac{\partial \hat{Q}[\underline{x},c]}{\partial x_j}$ implies a functional or variational partial derivative. (See for example Wang 1964.)

Equation (h.2.6) becomes with this substitution

$$\hat{Q}[\underline{x},c] = \min_{\underline{u}(y) \in U(\underline{u})} \left\{ \delta c \int_S G(s,c,\underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}) |J(s,c)| ds + \hat{Q}[\underline{x},c] + \delta c \int_S \sum_{j=1}^{n} \frac{\partial \hat{Q}[\underline{x},c]}{\partial x_j} \frac{\partial x_j}{\partial c} ds + \frac{\partial \hat{Q}[\underline{x},c]}{\partial c} \delta c \right\} + o(\delta c) \tag{h.2.8}$$

The terms $\hat{Q}[\underline{x},c]$ and $\dfrac{\partial \hat{Q}[\underline{x},c]}{\partial c} \delta c$ may be removed from the braces on the right hand side as they are independent of the control $\underline{u}(\underline{y})$. Cancelling $\hat{Q}[\underline{x},c]$, dividing by δc and letting $\delta c \to o$, then

$$\frac{-\partial \hat{Q}[\underline{x},c]}{\partial c} = \min_{\underline{u}(\underline{y}) \in U(\underline{u})} \int_S \left\{ G(s,c,\underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}) |J(s,c)| + \sum_{j=1}^{n} \frac{\partial \hat{Q}[\underline{x},c]}{\partial x_j} \frac{\partial x_j}{\partial c} \right\} ds \tag{h.2.9}$$

This result holds for all c. The optimal solution must satisfy (h.2.9) as well as (h.2.1). This yields a complete set of equations to determine $\hat{Q}[\underline{x},c]$ being minimized with respect to the 'initial' state $\underline{x}$.

Expression (h.2.9) is an extended form of Bellman's equation applicable for the distributed parameter problem formulated. Bellman's equation is equivalent to the Hamilton-Jacobi partial differential equation in the calculus of variations and is sometimes referred to as the Hamilton-Jacobi-Bellman equation; it is both necessary and sufficient for optimality. (See for example Lee 1964, Dreyfus 1965 among others for further discussion.) It is a functional partial differential equation in $\hat{Q}$, with initial conditions (from the definition of $\hat{Q}$, expressions (h.2.3, 3a, 3b)),

(h.2.9a) $\quad \hat{Q}[\underline{x},c^b] = o$

where c takes the value c^b on ∂Y^b, that is the region Y" over which G is integrated, vanishes.

The solution of the Hamilton-Jacobi-Bellman equation is, needless to say, very difficult in general. Methods for solution are available in works for example on classical mechanics and control theory. (See Dreyfus 1965.) Rozonoer (1959) outlines a heuristic approach to the solution. The optimal control follows from the solution of (h.2.9) for $\hat{Q}$.

As a result of the difficulties involved in solving the Hamilton-Jacobi -Bellman equation, it will be found more convenient to transform this equation into the lower order equations (Hamilton's canonical equations analogue) occurring in the maximum principle and the Hamiltonian form of the calculus of variations. (See for example Lee 1964 for discussion in the lumped parameter case.) Absolute minimality is replaced by relative minimality in so doing.

H.2.3 Transformation of equation (h.2.9). Although the original version of the maximum principle for lumped parameter systems as derived by Pontryagin et al (1962) was independent of Bellman's principle of optimality (dynamic programming), a relationship has since been shown. See for example Rozonoer (1959), Desoer (1961), Lee (1964), Dreyfus (1965), and Boltyanskii (1971) among others. Using these works as a guide, along with Long (1972), the distributed parameter form of Bellman's equation, equation (h.2.9), may be reinterpreted in the form of a maximum principle for the systems type II. This may be done by showing that the gradient vector of $\hat{Q}$ is related to the adjoint vector required in the canonical equations.

Recalling that $c = c(y_1, y_2, y_3)$, (h.2.9) becomes (dropping arguments)

$$- \sum_{i=1}^{3} \frac{\partial \hat{Q}}{\partial y_i} \frac{\partial y_i}{\partial c} = \min_{\underline{u} \in U} \int_S \left\{ G |J(s,c)| + \sum_{j=1}^{n} \sum_{i=1}^{3} \frac{\partial \hat{Q}}{\partial x_j} \frac{\partial x_j}{\partial y_i} \frac{\partial y_i}{\partial c} \right\} ds \tag{h.2.10a}$$

and further reduces, if the boundaries are aligned with the y_1, y_2 and y_3 axes, to

$$- \sum_{i=1}^{3} \frac{\partial \hat{Q}}{\partial y_i} = \min_{\underline{u} \in U} \int_S \left\{ G + \sum_{j=1}^{n} \sum_{i=1}^{3} \frac{\partial \hat{Q}}{\partial x_j} \frac{\partial x_j}{\partial y_i} \right\} ds \tag{h.2.10b}$$

The result (h.2.10b) is subject to a certain qualification, however, a qualification resulting from setting the Jacobian and the $\frac{\partial y_i}{\partial c}$ terms equal to unity in (h.2.10a). This simplification is only possible if the increments δy_i are the same in each of the coordinate directions implying that the inner and outer boundaries ∂Y^a and ∂Y^b are concentric cubes. Where the δy_i differ, ratio terms of the increments according to the particular problem would have to be incorporated. (h.2.10a) remains applicable in all cases. The result (h.2.10b) is nevertheless applicable for all planar regions (and three dimensional regions with two or three interval limits the same) with outer boundaries only by introducing a suitable imaginary inner boundary. For example, the inner boundary in the two dimensional case would correspond to a line parallel to the long side of the rectangle; boundary conditions on this inner boundary would be continuity conditions on the state across the boundary.

Exchange the minimization problem for a maximization problem according to max(-E) = - min(E) as follows

$$\text{(h.2.11)} \qquad \sum_{i=1}^{3} \frac{\partial \hat{Q}}{\partial y_i} = \max_{\underline{u} \in U} \int_S \left\{ G(-1) + \sum_{j=1}^{n} \sum_{i=1}^{3} \psi_j^i \, f_j^i \right\} ds$$

where the vectors $\underline{\psi}^i(\underline{y})$, i = 1, 2, 3 with n components defined by

$$\text{(h.2.12)} \qquad \underline{\psi}^i \triangleq \left(\frac{-\partial \hat{Q}}{\partial x_1}\bigg|^i, \ldots, \frac{-\partial \hat{Q}}{\partial x_n}\bigg|^i \right)^T \qquad i = 1, 2, 3$$

have been introduced. The superscript i denotes that the state is associated with system equation i, i = 1, 2, 3.

If then, a Hamiltonian, analogous to other sections, is defined as

$$\text{(h.2.13)} \qquad \begin{aligned} H(\underline{y}, \underline{x}, \ldots, \partial_{\ell}\underline{x}, \ldots, \underline{\psi}^i, \underline{u}) \triangleq\; & - G(\underline{y}, \underline{x}, \ldots, \partial_{\ell}\underline{x}, \ldots, \underline{u}) \\ & + \sum_{j=1}^{n} \sum_{i=1}^{3} \psi_j^i(\underline{y}) \, f_j^i(\underline{y}, \underline{x}, \ldots, \partial_{\ell}\underline{x}, \ldots, \underline{u}) \end{aligned}$$

then (h.2.11) becomes

$$\text{(h.2.14)} \qquad \sum_{i=1}^{3} \frac{\partial \hat{Q}}{\partial y_i} = \max_{\underline{u} \in U} \int_S H \, ds$$

Considering the arbitrariness of the location of the surface S, then (h.2.14) is a statement of the maximum principle of Pontryagin.

The result (h.2.14) implies that the control $\underline{u} \in U$ is chosen over the domain Y such that the Hamiltonian as given by (h.2.13) is maximized everywhere. H is a function of $\underline{u}$ through G and $\underline{f}^i$, i = 1, 2, 3, which contain $\underline{u}$. The vectors $\underline{\psi}^i$, i = 1, 2, 3 are obtained from the partial derivatives of $\hat{Q}$ with respect to the state $\underline{x}$ as given in (h.2.11). This may be a difficult task, first finding $\hat{Q}$, and so an alternative means of deriving $\underline{\psi}^i$ would be desirable. This is achieved by deriving a set of adjoint equations in $\underline{\psi}^i$ analogous to those in §F and §J.

From (h.2.12), $\underline{\psi}^i$ is a function of $\underline{x}$, but in the functional form (h.2.13), $\underline{x}$, ... , $\partial_{\underline{\ell}}\underline{x}$, ... , $\underline{u}$, $\underline{\psi}^i$, and $\underline{y}$ may be regarded as independent quantities. Then

$$\text{(h.2.15)} \qquad \frac{\partial x_j}{\partial y_i} = \frac{\partial H}{\partial \psi_j^i} \qquad\qquad \begin{array}{l} j = 1, \ldots, n \\ i = 1, 2, 3 \end{array}$$

This is an alternative representation of (h.2.1). Boundary conditions (h.2.1a) apply.

Assume now that the end state conditions (h.2.1a) are given a more general form (§C.3) relating to a set of states $(S^a)^i$ and $(S^b)^i$, $i = 1, 2, 3$ where the boundaries are aligned with the y_1, y_2 and y_3 directions. Envisaging a related set of equations to (h.2.15) in ψ_j^i (adjoint equations), the boundary conditions for these may be obtained from (h.2.9a). Isolating the behaviour over one spatial coordinate y_i for discussion,

$$\text{(h.2.9b)} \qquad \hat{Q}[\underline{x}, \ldots, y_i^b] = 0 \qquad\qquad \forall\ \underline{x} \in (S^b)^i$$

where $c^b = y_i^b$ for the portion of ∂Y^b orthogonal to the y_i axis. If it is assumed that $(S^b)^i$ has a tangent and normal at each point; then (h.2.9b) may be regarded as the equation of $(S^b)^i$ and so the normal to $(S^b)^i$ is given by

$$\left(\left.\frac{\partial \hat{Q}}{\partial x_1}\right|^i, \ldots, \left.\frac{\partial \hat{Q}}{\partial x_n}\right|^i \right)_{y_i^b}$$

By (h.2.12), this is $-\underline{\psi}^i(\ldots, y_i^b)$. This is the transversality condition at y_i^b; namely $\underline{\psi}^i$ is normal to $(S^b)^i$.

If $(S^b)^i$ is not prescribed at all, then $\underline{\psi}^i$ is normal to all vectors, and $\underline{\psi}^i(\ldots, y_i^b) = o$. If $(S^b)^i$ is a single point, then $\underline{x}(\ldots, y_i^b)$ is fixed and there is no condition on $\underline{\psi}^i(\ldots, y_i^b)$; that is $(S^b)^i$ has no normal. If some $x_j(\ldots, y_i^b)$ are prescribed, then the ψ_j^i corresponding to the remaining x_j are all zero at y_i^b. $\underline{Q}[\underline{x}, \ldots, y_i^b]=o$ identically in these remaining $x_j \Rightarrow \psi_j^i = -\dfrac{\partial\underline{Q}}{\partial x_j} = o.$

In general, suppose $(S^b)^i$ is the intersection of smooth surfaces $(S_\beta^b)^i = o$, $\beta = 1, \ldots, q$, (§C.3), with a unique tangent plane T at each point. Let $\underline{\eta}$ be a vector in T. Then $\underline{\eta}$ is orthogonal to the normal to each $(S_\beta^b)^i$

$$\sum_{j=1}^{n} \frac{\partial (S_\beta^b)^i}{\partial x_j} \eta_j = o \qquad\qquad \beta = 1, \ldots, q$$

These equations can be solved for q components of $\underline{\eta}$ in terms of the remaining n-q arbitrary components. Then in

$$\sum_{j=1}^{n} \psi_j^i \eta_j = o$$

the coefficients of the arbitrary components of $\underline{\eta}$ are all zero giving n-q conditions involving ψ_j^i and x_j at y_i^b (see for example Leitmann 1966, pp 21-23).

For the boundary conditions at y_i^a, $\underline{Q}[\underline{x}, \ldots, y_i^a]$ may be regarded as the equation of $(S^a)^i$. For deviations $\underline{\delta x}$ from the state $\underline{x}$ corresponding to the optimal solution on $\underline{Q}[\underline{x}, \ldots, y_i^a]$

$$Q[\underline{x} + \underline{\delta x}, \ldots, y_i^a] \geq \underline{Q}[\underline{x}, \ldots, y_i^a]$$

Expanding the left hand side about $\underline{Q}[\underline{x}, \ldots, y_i^a]$

$$\sum_{j=1}^{n} \left.\frac{\partial\underline{Q}}{\partial x_j}\right|^i \delta x_j + o(\delta x_j^2) \geq o$$

Substituting ψ_j^i and for δx_j small

$$\sum_{j=1}^{n} \psi_j^i \, \delta x_j \leq o$$

Now for some δx_j (any vector in the tangent plane), suppose $\sum_{j=1}^{n} \psi_j^i \; \delta x_j < o$.

If δx_j is replaced with $-\delta x_j$ (for interior points of $\hat{Q}$ only), then $\sum_{j=1}^{n} \psi_j^i \delta x_j > o$. This can only be true if $\sum_{j=1}^{n} \psi_j^i \delta x_j = o$. This is the required transversality condition; namely $\underline{\psi}$ is orthogonal to $(S^a)^i$.

From (h.2.14), for fixed $\underline{x}$ on an optimal solution, H is to be maximized with respect to $\underline{u}$ and this process determines the minimizing $\underline{\hat{u}}$ for this $\underline{x}$ at a fixed $\underline{y}$. Write

$$\text{(h.2.16)} \qquad \begin{aligned} &\max_{\underline{u} \in U} \int_S H(\underline{y}, \underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{\psi}^i, \underline{u})\, ds \\ &= H^*(\underline{y}, \underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{\psi}^i, \underline{u}) \Big|_{\underline{\hat{u}}} \end{aligned}$$

For varying $\underline{x}(\underline{y})$ along an optimal solution, the minimizing $\underline{\hat{u}}$ is obtained as a function of $\underline{x}$ and so $\underline{\hat{u}} = \underline{\hat{u}}[\underline{x}(\underline{y})]$. Also $\underline{\psi}^i = \underline{\psi}^i[\underline{x}(\underline{y})]$ and (h.2.16) becomes

$$\text{(h.2.17)} \qquad \begin{aligned} &\max_{\underline{u} \in U} \int_S H(\underline{y}, \underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{\psi}^i, \underline{u})\, ds \\ &= H^*(\underline{y}, \underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{\psi}^i, \underline{u}) \Bigg|_{\substack{\underline{u} = \underline{\hat{u}}[\underline{x}(\underline{y})] \\ \underline{\psi}^i = \underline{\psi}^i[\underline{x}(\underline{y})]}} \\ &= \hat{H}(\underline{x}, \underline{y}) \end{aligned}$$

From (h.2.14)

$$\text{(h.2.18)} \qquad \hat{H}(\underline{x},y) = \sum_{i=1}^{3} \frac{\partial \hat{Q}}{\partial y_i}$$

Now, differentiating (h.2.12) and assuming that $\hat{Q}$ is twice (partially) differentiable with respect to all the states

$$\sum_{i=1}^{3} \frac{\partial \psi_j^i}{\partial y_i} = \sum_{i=1}^{3} \frac{\partial}{\partial y_i} \left(- \frac{\partial \hat{Q}}{\partial x_j} \bigg|^{i} \right)$$

$$\text{(h.2.19)} \qquad = - \sum_{i=1}^{3} \left\{ \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left[\frac{\partial \hat{Q}}{\partial x_j} \right]^{i} \frac{\partial x_k}{\partial y_i} + \frac{\partial^2 \hat{Q}}{\partial y_i \partial x_j} \bigg|^{i} \right\}$$

$$= \sum_{i=1}^{3} \sum_{k=1}^{n} \frac{\partial \psi_j^i}{\partial x_k} f_k^i - \frac{\partial \hat{H}}{\partial x_j}$$

along an optimal solution, using (h.2.18).

(a) First suppose $\underline{u}$ is unconstrained. Then $\hat{\underline{u}}$ must satisfy the necessary conditions for maximizing H, namely

$$\text{(h.2.20)} \qquad \int_S \frac{\partial H}{\partial \underline{u}} \, ds \bigg|_{\hat{\underline{u}}} = 0$$

assuming these derivatives exist.

Using (h.2.17), for variations δx_j

$$\delta \hat{H} = \int_S \left[\delta x_j \frac{\partial H}{\partial x_j} + \delta x_j \frac{\partial H}{\partial \underline{u}} \frac{\partial \underline{u}}{\partial x_j} + \sum_{i=1}^{3} \delta x_j \frac{\partial H}{\partial \underline{\psi}^i} \frac{\partial \underline{\psi}^i}{\partial x_j} \right.$$

$$+ \delta[\partial_{\underline{\ell}} x_j] \left.\frac{\partial H}{\partial[\partial_{\underline{\ell}} x_j]}\right] ds \Bigg|_{\underline{u} = \hat{\underline{u}}(\underline{x})}$$

$$= \int_S \Bigg[\delta x_j \frac{\partial H}{\partial x_j} + \sum_{i=1}^{3} \delta x_j \underline{f}^i \frac{\partial \underline{\psi}^i}{\partial x_j}$$

$$+ (-1)^{\underline{\ell}} \delta x_j \, \partial_{\underline{\ell}} \left[\frac{\partial H}{\partial[\partial_{\underline{\ell}} x_j]}\right]$$

$$+ (-1)^{\underline{\ell}-1} \partial_1^T \left\{ (\delta x_j)\, \partial_{\underline{\ell}-1} \left[\frac{\partial H}{\partial[\partial_{\underline{\ell}} x_j]}\right]\right\}$$

(h.2.21) $$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$+ \partial_1^T \left\{ (\partial_{\underline{\ell}-3} \delta x_j)\; \partial_2 \left[\frac{\partial H}{\partial[\partial_{\underline{\ell}} x_j]}\right]\right\}$$

$$- \partial_1^T \left\{ (\partial_{\underline{\ell}-2} \delta x_j)\; \partial_1 \left[\frac{\partial H}{\partial[\partial_{\underline{\ell}} x_j]}\right]\right\}$$

$$+ \partial_1^T \left\{ (\partial_{\underline{\ell}-1} \delta x_j) \left[\frac{\partial H}{\partial[\partial_{\underline{\ell}} x_j]}\right]\right\} \Bigg] ds \Bigg|_{\underline{u} = \hat{\underline{u}}(\underline{x})}$$

On the right hand side of (h.2.21), the last three terms and the terms understood with the usage of the ellipsis dots may be integrated. Noting that the location of S is arbitrary, then invoking the end-state conditions (h.2.1a), some of these terms are identically zero while the optimal solution will be required to be of a form such that the remainder of these terms vanish. The term preceding the ellipsis dots also may be set equal to zero in correspondence with the boundary conditions on $\underline{\psi}^i$; namely $\underline{\psi}^i$ is normal to the tangent plane of $(S^b)^i$. With δx_j lying in the tangent plane, ψ_j^i is chosen as

$$\text{(h.2.22)} \qquad \psi_j^i = (-1)^{\underline{\ell}-1}\partial_{\underline{\ell}-1}\left[\frac{\partial H}{\partial[\partial_{\underline{\ell}}x_j]}\right]$$

on the boundary and the product $\delta x_j \psi_j^i$ equals zero. (h.2.21) reduces to

$$\delta\hat{H} = \left[\delta x_j \frac{\partial H^*}{\partial x_j} + \sum_{i=1}^{3} \delta x_j \, \underline{f}^i \frac{\partial \underline{\psi}^i}{\partial x_j} + (-1)^{\underline{\ell}}\delta x_j \, \partial_{\underline{\ell}}\left[\frac{\partial H^*}{\partial[\partial_{\underline{\ell}}x_j]}\right]\right]_{\underline{u} = \underline{\hat{u}}(\underline{x})}$$

Dividing by δx_j and letting $\delta x_j \to o$

$$\text{(h.2.23)} \qquad \frac{\partial\hat{H}}{\partial x_j} = \left[\frac{\partial H^*}{\partial x_j} + \sum_{i=1}^{3} \underline{f}^i \frac{\partial\underline{\psi}^i}{\partial x_j} + (-1)^{\underline{\ell}}\partial_{\underline{\ell}}\left[\frac{\partial H^*}{\partial[\partial_{\underline{\ell}}x_j]}\right]\right]_{\underline{u} = \underline{\hat{u}}(\underline{x})}$$

Put (h.2.23) into (h.2.19) to give

$$\text{(h.2.24)} \qquad \sum_{i=1}^{3} \frac{\partial\psi_j^i}{\partial y_i} = -\frac{\partial H^*}{\partial x_j} - (-1)^{\underline{\ell}}\partial_{\underline{\ell}}\left[\frac{\partial H^*}{\partial[\partial_{\underline{\ell}}x_j]}\right]\Bigg|_{\underline{u} = \underline{\hat{u}}(\underline{x})} \qquad j = 1, \ldots, n$$

Wang (1964) and Brogan (1968b) among others use the result that if the solution of (h.2.18) is analytic (or regular, that is if it is defined and has a derivative at all points) then

$$\text{(h.2.24)'} \qquad \sum_{i=1}^{3} \frac{\partial\psi_j^i}{\partial y_i} = -\frac{\partial H}{\partial x_j} - (-1)^{\underline{\ell}}\partial_{\underline{\ell}}\left[\frac{\partial H}{\partial[\partial_{\underline{\ell}}x_j]}\right]\Bigg|_{\underline{u} = \underline{\hat{u}}(\underline{x})} \qquad j = 1, \ldots, n$$

This is the required equation in $\underline{\psi}^i$ (adjoint equation). It determines $\underline{\psi}^i$ for all $\underline{y}$. Since H is linear in ψ_j^i, the equation is linear in ψ_j^i.

Boundary conditions for (h.2.24)' were discussed above.

(b) Suppose the range of $\underline{u}$ is restricted by constraints of the form (§D.2)

(h.2.25) $$h_\alpha[\underline{y}, \underline{u}(\underline{y})] = o \qquad \alpha = 1, 2, \ldots, m$$

Then any admissible $\underline{u}$, optimal or not, must satisfy (h.2.25). Write

(h.2.26)
$$\begin{aligned}\bar{H} &= H + \sum_{\alpha=1}^{m} \mu_\alpha h_\alpha \\ &= -G(\underline{y}, \underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}) \\ &\quad + \sum_{j=1}^{n} \sum_{i=1}^{3} \psi_j^i(\underline{y})\, f_j^i(\underline{y}, \underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}) \\ &\quad + \sum_{\alpha=1}^{m} \mu_\alpha(\underline{y}) h_\alpha(\underline{y}, \underline{u})\end{aligned}$$

From the theory of Lagrange multipliers, a necessary condition for H to be maximized at $\underline{u} = \hat{\underline{u}}$ subject to $h_\alpha = o$ is that $\hat{\underline{u}}$ satisfy

(h.2.27) $$\int_S \frac{\partial \bar{H}}{\partial \underline{u}}\, ds = o$$

Possible values of $\underline{u}$ are found by solving (h.2.25) and (h.2.27) simultaneously. Then

$$\hat{H}(\underline{x}, \underline{y}) = \max_{\underline{u} \in U} \int_S H\, ds \qquad \text{subject to } h_\alpha = o$$

(h.2.28) $$= \max_{\underline{u} \in U} \int_S \bar{H}\, ds \qquad \text{subject to } h_\alpha = o$$

(h.2.29) $$= \hat{\bar{H}}(\underline{x},\underline{y}) \qquad \forall\ \underline{x},\ \underline{y} \text{ on the optimal solution.}$$

Note that $\hat{\bar{H}}$ is derived from $\bar{H}$ by taking $\underline{x}$ on the optimal solution and putting $\underline{u} = \hat{\underline{u}}(\underline{x})$, $\underline{\psi}^i = \underline{\psi}^i(\underline{x})$. Also $\underline{\mu}$ depends on $\underline{x}$ and so $\underline{\mu} = \underline{\mu}(\underline{x})$.

Since $\hat{H} = \hat{\bar{H}}$ for all $\underline{x}$ on the optimal solution,

$$\frac{\partial \hat{H}}{\partial x_j} = \frac{\partial \hat{\bar{H}}}{\partial x_j} \quad \text{for all } \underline{x} \text{ on the optimal solution,}$$

(h.2.30)
$$= \left[\frac{\partial \bar{H}^*}{\partial x_j} + \frac{\partial \bar{H}^*}{\partial \underline{u}} \frac{\partial \underline{u}}{\partial x_j} + \sum_{i=1}^{3} \frac{\partial \bar{H}^*}{\partial \underline{\psi}^i} \frac{\partial \underline{\psi}^i}{\partial x_j} + \frac{\partial \bar{H}^*}{\partial \underline{\mu}} \frac{\partial \underline{\mu}}{\partial x_j} + (-1)^{\underline{\ell}} \partial_{\underline{\ell}} \left[\frac{\partial \bar{H}^*}{\partial [\partial_{\underline{\ell}} x_j]} \right] \right]_{\underline{u} = \hat{\underline{u}}(\underline{x})}$$

Now $\dfrac{\partial \bar{H}^*}{\partial \underline{\mu}} = \underline{h} = o$ for $\underline{u} = \hat{\underline{u}}$. Also in $\bar{H}$, $\underline{h}$ is independent of

$\underline{x}$, ... , $\partial_{\underline{\ell}}\underline{x}$, ... , and $\underline{\psi}^i$ and so

$$\frac{\partial \bar{H}^*}{\partial x_j} = \frac{\partial H^*}{\partial x_j}, \quad \frac{\partial \bar{H}^*}{\partial [\partial_{\underline{\ell}} \underline{x}]} = \frac{\partial H^*}{\partial [\partial_{\underline{\ell}} \underline{x}]}, \quad \frac{\partial \bar{H}^*}{\partial \underline{\psi}^i} = \frac{\partial H^*}{\partial \underline{\psi}^i}$$

That is

$$\frac{\partial \hat{H}}{\partial x_j} = \frac{\partial H^*}{\partial x_j} + \sum_{i=1}^{3} \underline{f}^i \frac{\partial \underline{\psi}^i}{\partial x_j} + (-1)^{\underline{\ell}} \partial_{\underline{\ell}} \left[\frac{\partial H^*}{\partial [\partial_{\underline{\ell}} x_j]} \right] \Bigg|_{\underline{u} = \hat{\underline{u}}(\underline{x})}$$

as before, leading to (h.2.24)' again.

(c) Suppose $\underline{u}$ is subject to inequality constraints (§D.2.1)

$$h_\alpha[\underline{y},\underline{u}(\underline{y})] \leq o \qquad \alpha = 1, \ldots, m \tag{h.2.31}$$

Write

$$h_\alpha[\underline{y},\underline{u}] + (u^*_\alpha)^2 = o \tag{h.2.32}$$

where the u^*_α are regarded as additional control variables. Then (h.2.32) is of the form (h.2.25) of item (b) and again gives the result (h.2.24)'.

<u>H.2.4 Summary of the optimality conditions.</u> For a given system model of type II defined in an orthogonal domain Y with coordinates $\underline{y} = (y_1, y_2, y_3)^T$ by the state (system) equations

$$\frac{\partial x_j}{\partial y_i} = f^i_j[\underline{y}, \underline{x}, \ldots, \partial_{\underline{\ell}}\underline{x}, \ldots, \underline{u}] \qquad \begin{array}{l} i = 1, 2, 3 \\ j = 1, \ldots, n \end{array} \tag{h.2.1}$$

$\underline{\ell} = (\ell_h, \ell_k)$; (h, k=1, 2, 3 $\neq$ i), with end-state conditions of the form

(h.2.1a) $\quad \partial_{\underline{\ell}-1}x_j$ specified at boundaries y^a_i, y^b_i, i = 1, 2, 3

an admissible control $\underline{u} \in U$ for all $\underline{y} \in Y$ is to be chosen so as to minimize

$$Q = \int_Y G[\underline{y}, \underline{x}, \ldots, \partial_{\underline{\ell}}\underline{x}, \ldots, \underline{u}]\, d\underline{y} \tag{h.2.3}$$

where $\underline{f}^i$, $\underline{x}$ and $\underline{u}$ are n, n and r component vectors respectively.

In order for the control to be optimal, three n-component vectors $\underline{\psi}^i(\underline{y}) = (\psi^i_1, \ldots, \psi^i_n)^T$, i = 1, 2, 3 are introduced satisfying the (adjoint) equations,

$$\sum_{i=1}^{3} \frac{\partial \psi^i_j}{\partial y_i} = -\frac{\partial H}{\partial x_j} - (-1)^{\underline{\ell}}\partial_{\underline{\ell}}\left[\frac{\partial H}{\partial[\partial_{\underline{\ell}} x_j]}\right] \qquad j = 1, \ldots, n \tag{h.2.24)'}$$

with natural boundary conditions

$$x_j \text{ given or}$$

(h.2.22)
$$-\psi_j^i + (-1)^{\underline{\ell}-1}\partial_{\underline{\ell}-1}\left[\frac{\partial H}{\partial[\partial_{\underline{\ell}} x_j]}\right] = o$$

at y_i^a, y_i^b, $i = 1, 2, 3$

in order that the Hamiltonian

(h.2.13)
$$H(\underline{y}, \underline{x}, \ldots, \partial_{\underline{\ell}}\underline{x}, \ldots, \underline{\psi}^i, \underline{u}) = -G(\underline{y}, \underline{x}, \ldots, \partial_{\underline{\ell}}\underline{x}, \ldots, \underline{u}) + \sum_{i=1}^{3}\sum_{j=1}^{n} \psi_j^i(\underline{y})\, f_j^i(\underline{y}, \underline{x}, \ldots, \partial_{\underline{\ell}}\underline{x}, \ldots, \underline{u})$$

takes on a maximum value with respect to $\underline{u}$. State continuity conditions over an imaginary inner boundary may also apply.

H.2.5 Boundary criterion. Suppose that the problem is as before except that the criterion is (§D.3)

(h.2.33)
$$Q = \int_Y G(\underline{y}, \underline{x}, \ldots, \partial_{\underline{\ell}}\underline{x}, \ldots, \underline{u})d\underline{y} + \int_{\partial Y^b} g(\underline{x})\; dy_1 dy_3 \Big|_{y_2^b}$$

Additional terms may be similarly appended for the boundary $\partial Y^b|_{y_1^b}$ $\partial Y^b|_{y_3^b}$ and ∂Y^a. Also for simplicity only consider the region Y defined with side intervals $[y_1^a, y_1^b]$, $[y_2^a, y_2^b]$ and $[y_3^a, y_3^b]$.

Introduce extra state and control variables x_{n+1}, u_{r+1}, u_{r+2} and u_{r+3} related by

(h.2.34)
$$\frac{\partial x_{n+1}}{\partial y_i} = u_{r+i} = f_{n+1}^i \qquad i = 1, 2, 3$$

with $x_{n+1}(\ \ldots\ ,\ y_i^a) = o$, $i = 1, 2, 3$ and a new end criterion

$$(S_{q+1}^b)^2 = g(\underline{x}) - x_{n+1}\Big|_{y_2^b} = o \tag{h.2.35}$$

Then

$$\int_{y_3}\int_{y_1} g(\underline{x})\,dy_1 dy_3\Big|_{y_2^b} = \int_{y_3}\int_{y_1} x_{n+1}(y_1,\ y_2^b\ ,\ y_3)\ dy_1 dy_3$$

$$= \int_{y_3}\int_{y_1}\left\{x_{n+1}(y_1,\ y_2^b\ ,\ y_3) - x_{n+1}(y_1,\ y_2^a\ ,\ y_3)\right\} dy_1 dy_3 \tag{h.2.36}$$

$$= \int_{y_2}\int_{y_3}\int_{y_1} u_{r+2}\ dy_1 dy_3 dy_2$$

from (h.2.35) and (h.2.34).

This implies using a new criterion

$$\tilde{Q} = \int_Y \left\{G(\underline{y},\ \underline{x},\ \ldots\ ,\ \partial_{\underline{\ell}}\underline{x},\ \ldots\ ,\ \underline{u}) + u_{r+2}\right\} d\underline{y}$$
$$= \int \tilde{G}\ d\underline{y} \tag{h.2.37}$$

The previous method now applies if extra adjoint variables ψ_{n+1}^i, $i = 1, 2, 3$ are introduced. Define

$$\tilde{H} = H + (-1)u_{r+2} + \psi_{n+1}^1 u_{r+1} + \psi_{n+1}^2 u_{r+2} + \psi_{n+1}^3 u_{r+3} \tag{h.2.38}$$

where H is independent of x_{n+1}, u_{r+i} and ψ_{n+1}^i, $i = 1, 2, 3$.

Then

$$\sum_{i=1}^{3} \frac{\partial\psi_j^i}{\partial y_i} = -\frac{\partial H}{\partial x_j} - (-1)^{\underline{\ell}}\partial_{\underline{\ell}}\left[\frac{\partial H}{\partial[\partial_{\underline{\ell}} x_j]}\right] \qquad j = 1, \ldots, n$$

(h.2.39)

$$\text{as before and } \sum_{i=1}^{3} \frac{\partial\psi_{n+1}^i}{\partial y_i} = o$$

$\Rightarrow \psi_{n+1}^i$ is a constant in the y_i direction.

$\tilde{H}$ is to be maximized with respect to u_{r+i}, $i = 1, 2, 3$, and since these controls are unconstrained

$$\frac{\partial\tilde{H}}{\partial u_{r+i}} = o \qquad i = 1, 2, 3$$

Therefore

$$-1 + \psi_{n+1}^2 = o$$

(h.2.40)

$$\psi_{n+1}^1 = o$$

$$\psi_{n+1}^3 = o$$

and also $\tilde{H} = H$, all y_1, y_2.

The new transversality conditions: With $\underline{\eta}$ as before, that is a vector lying in the tangent plane to each surface $(S_\beta^b)^2$, $\beta = 1, \ldots, q+1$, let $\underset{\sim}{\eta} = (\eta_1, \ldots, \eta_n, \eta_{n+1})^T$.

Then

(h.2.41)

$$\sum_{j=1}^{n+1} \frac{\partial(S_\beta^b)^2}{\partial x_j}\eta_j = o \qquad \beta = 1, \ldots, q+1$$

That is

$$\sum_{j=1}^{n} \frac{\partial g}{\partial x_j} \eta_j - \eta_{n+1} = o \tag{h.2.42}$$

Also $\underset{\sim}{\psi}^2 = (\psi_1^2, \ldots, \psi_n^2, \psi_{n+1}^2)^T$ is orthogonal to $\underset{\sim}{\eta}$ and so

$$\sum_{j=1}^{n} \psi_j^2 \eta_j + \psi_{n+1}^2 \eta_{n+1} = o \tag{h.2.43}$$

Therefore, combining (h.2.42) and (h.2.43), and using (h.2.40), the transversality conditions are

$$\sum_{j=1}^{n} \left(\psi_j^2 + \frac{\partial g}{\partial x_j} \right) \eta_j = o \tag{h.2.44}$$

along with (h.2.41) for $\beta = 1, \ldots, q$.

Similar results were obtained by Lurie (1963) using the calculus of variations arguments.

H.3 DISCUSSION

Using an imbedding procedure (whereby the original problem is replaced by a sequence of smaller problems) in conjunction with the principle of optimality, dynamic programming reduces the optimal control problem to the solution of the Hamilton-Jacobi-Bellman functional equation. The underlying restrictive assumption related to smoothness and continuity conditions on $\hat{Q}$. (Compare with the classical calculus of variations approach in §J which requires the alternative assumption of free variations. Rozonoer's approach in §F requires neither restrictive assumption.) The conditions arose out of the solution technique and not from the essence of the problem. The relationship of this functional equation to a distributed parameter version of Pontryagin's maximum principle was shown by relating the gradient vector of $\hat{Q}$ to the adjoint vector characteristic of the canonical equations.

The conditions for optimality assume the form of simultaneous differential equations which, for the lumped parameter case, have an analogous mathematical structure to Hamilton's canonical equations of classical mechanics (see article §A.2, equation (a.2.6) and Rozonoer 1959). It is only an analogy (and hence the usage of the terminology 'Hamilton's' equations in control theory) and no physical significance should be attached to it. (The analogy arises if the generalized coordinates are set equal to the state, and momenta set equal to the adjoint variables. Control variables are explicitly introduced in control theory but are eliminated in analytical mechanics.) As noted by Rozonoer (1959), the relationship in control theory between dynamic programming and the maximum principle is analogous to the relationship in mechanics between the Hamilton-Jacobi equations and the Hamilton canonical equations if the preceding note in parenthesis is adhered to. They are alternative ways of characterizing the optimal control problem or system behaviour respectively. See also Lanczos (1949).

It may also be recalled from §A.2 that the state space concepts of control theory were a generalisation of phase space concepts of classical dynamics. This generalisation and the previously mentioned analogy (of Hamilton's equations) were two separate introductions to control theory and no connection should be attempted.

In the above, constraints defining the admissible set U were considered to be independent of values taken by the state. The maximum principle so derived is only valid in this context. (However Bellman's functional equation obtained en route to the maximum principle is valid for all forms of constraints.) Further conditions relating to discontinuities in the adjoint variables are necessary before the above maximum principle can be generalised. (See Rozonoer 1959 for comments.)

Two different philosophies of approach are given in §F and §J for the determination of conditions for optimality of a design. Although the approaches are applied to different system types their use is not exclusive to any particular system type but the approaches are in a sense complementary in that they all reduce to related optimality conditions (maximum principles) and are interchangeable to a large degree. The usefulness of adopting several approaches is apparent not only in the unification but also in the slightly different interpretations obtained. Observations regarding the characteristics, computational treatment and application potential of the related optimality conditions of system type I, as are outlined in §F.4, are valid here.

The conditions for type II systems presented here, combined with the conditions of §F for type I and §J for type III systems, will cover a very broad class of problems likely to be encountered in structural design. The choice of the class of system with which to model any structure will vary with the characteristics of the particular structure. The present system model and type III model are suitable for systems behaving similarly in each of their independent variable directions. Where behaviour differs in one particular direction, the previous model (type I) would probably be preferred. To enable a comparison of the modelling and solution procedures involved with system types I, II and III, an illustration of the usage of the present conditions for systems type II may be found in the following section. The illustration problem is the same as that treated in the previous section for systems type I and in §K for type III.

§I A DESIGN ILLUSTRATION: SYSTEM MODEL TYPE II - SENSITIVITY DISCUSSION

I.1 GENERAL

For comparative purposes, the structural system of the problem in §G (and §K) is interpreted as a system model type II. From the symmetry of the problem and the similar expected behaviours in each of the independent variable directions, a system model of type II would appear to offer an attractive approach. A comparison with the formulation of the illustration in §G (and §K) is given and as may be expected, it is shown that all system models (I, II and III) and their associated optimality conditions (§F, §H and §J respectively), reduce to the same set of equations to solve for the optimal solution to be gained.

Several reductions are possible for this structural system (because of its symmetry) to a system model type II. Thus by suitable modifications of the choice of state variables, alternative type II forms evolve. One such alternative is emphasized as it coincides with the model in the fundamental study of Lurie (1963); that is it appears without state derivatives on the right hand side (but also many of the state variables lose their physical significance). It is also shown to reduce to the same result for the optimal solution.

To complete the solution for this section and sections §G and §K, the deviation of the system behaviour and properties from the optimal caused by deviations in the system parameters are examined. This forms the rudiments of a sensitivity analysis of the problem. It is shown that sensitivity considerations can be included as an integral part of system design.

The problem in §G (and §K) is that of the minimal weight of a freely vibrating plate where a constraint on the fundamental frequency of a reference plate is imposed. The governing plate equation (g.2.3) reads

$$\frac{\partial^2}{\partial y_1^2}\left[D\,\frac{\partial^2 W}{\partial y_1^2} + D\nu\,\frac{\partial^2 W}{\partial y_2^2}\right] + \frac{\partial^2}{\partial y_2^2}\left[D\,\frac{\partial^2 W}{\partial y_2^2} + D\nu\,\frac{\partial^2 W}{\partial y_1^2}\right]$$

(i.1.1)

$$+\,2\,\frac{\partial^2}{\partial y_1\partial y_2}\left[D(1-\nu)\,\frac{\partial^2 W}{\partial y_1\partial y_2}\right] - e^2 D^{\frac{1}{3}} W = 0$$

with boundary conditions (g.2.8)

$$W\Big|_{y_1=0,a} = 0 \qquad D\frac{\partial^2 W}{\partial y_1^2} + D\nu\frac{\partial^2 W}{\partial y_2^2}\Big|_{y_1=0,a} = 0$$

(i.1.1a)

$$W\Big|_{y_2=0,b} = 0 \qquad D\frac{\partial^2 W}{\partial y_2^2} + D\nu\frac{\partial^2 W}{\partial y_1^2}\Big|_{y_2=0,b} = 0$$

and criterion (g.2.5)

(i.1.2) $$Q = \int_{y_2}\int_{y_1} D^{\frac{1}{3}}\, dy_1\, dy_2$$

The symbols retain the same meaning as in §G.2.

I.2 STATE-CONTROL INTERPRETATION

A set of state variables $\{x_j;\ j=1, \ldots, n\}$ are introduced subscripted for convenience in increasing order of the derivative of W involved. For any derivative of order α there are introduced $\alpha+1$ state variables. Set

$$x_1 \triangleq W$$

$$x_2 \triangleq \frac{\partial W}{\partial y_1}$$

$$x_3 \triangleq \frac{\partial W}{\partial y_2}$$

$$x_4 \triangleq D\frac{\partial^2 W}{\partial y_1^2} + D\nu\frac{\partial^2 W}{\partial y_2^2}$$

$$x_5 \triangleq D(1-\nu)\frac{\partial^2 W}{\partial y_1 \partial y_2}$$

(i.2.1) $$x_6 \triangleq D\frac{\partial^2 W}{\partial y_2^2} + D\nu\frac{\partial^2 W}{\partial y_1^2}$$

$$x_7 \triangleq \frac{\partial}{\partial y_2}\left[D\frac{\partial^2 W}{\partial y_1^2} + D\nu\frac{\partial^2 W}{\partial y_2^2}\right]$$

$$x_8 \triangleq \frac{\partial}{\partial y_1}\left[D\frac{\partial^2 W}{\partial y_1^2} + D\nu\frac{\partial^2 W}{\partial y_2^2}\right] + \frac{\partial}{\partial y_2}\left[D(1-\nu)\frac{\partial^2 W}{\partial y_1 \partial y_2}\right]$$

$$x_9 \hat{=} \frac{\partial}{\partial y_2}\left[D \frac{\partial^2 w}{\partial y_2^2} + D\nu \frac{\partial^2 w}{\partial y_1^2}\right] + \frac{\partial}{\partial y_1}\left[D(1-\nu) \frac{\partial^2 w}{\partial y_1 \partial y_2}\right]$$

$$x_{10} \hat{=} \frac{\partial}{\partial y_1}\left[D \frac{\partial^2 w}{\partial y_2^2} + D\nu \frac{\partial^2 w}{\partial y_1^2}\right]$$

The control, $u \hat{=} D$.

In addition, certain equation consistency terms or auxiliary dependent (control) variables are required such that when the state variables are differentiated, the resulting sets of equations are consistent with the original fourth order equation. That is their role is essentially one of giving the total order to the set of equations. They may be treated as auxiliary equation controls (see § C.3). Set

$$u_1 \hat{=} \frac{\partial}{\partial y_1}\left\{\frac{\partial}{\partial y_2}\left[D \frac{\partial^2 w}{\partial y_1^2} + D\nu \frac{\partial^2 w}{\partial y_2^2}\right]\right\}$$

$$u_2 \hat{=} \frac{\partial}{\partial y_2}\left\{\frac{\partial}{\partial y_2}\left[D \frac{\partial^2 w}{\partial y_1^2} + D\nu \frac{\partial^2 w}{\partial y_2^2}\right]\right\}$$

$$u_3 \hat{=} \frac{\partial}{\partial y_1}\left\{\frac{\partial}{\partial y_1}\left[D \frac{\partial^2 w}{\partial y_1^2} + D\nu \frac{\partial^2 w}{\partial y_2^2}\right] + \frac{\partial}{\partial y_2}\left[D(1-\nu) \frac{\partial^2 w}{\partial y_1 \partial y_2}\right]\right\}$$

$$u_4 \hat{=} \frac{\partial}{\partial y_2}\left\{\frac{\partial}{\partial y_1}\left[D \frac{\partial^2 w}{\partial y_1^2} + D\nu \frac{\partial^2 w}{\partial y_2^2}\right] + \frac{\partial}{\partial y_2}\left[D(1-\nu) \frac{\partial^2 w}{\partial y_1 \partial y_2}\right]\right\} \quad \text{(i.2.2)}$$

$$u_5 \hat{=} \frac{\partial}{\partial y_1}\left\{\frac{\partial}{\partial y_2}\left[D \frac{\partial^2 w}{\partial y_2^2} + D\nu \frac{\partial^2 w}{\partial y_1^2}\right] + \frac{\partial}{\partial y_1}\left[D(1-\nu) \frac{\partial^2 w}{\partial y_1 \partial y_2}\right]\right\}$$

$$u_6 \hat{=} \frac{\partial}{\partial y_1}\left\{\frac{\partial}{\partial y_1}\left[D \frac{\partial^2 w}{\partial y_2^2} + D\nu \frac{\partial^2 w}{\partial y_1^2}\right]\right\}$$

$$u_7 \hat{=} \frac{\partial}{\partial y_2}\left\{\frac{\partial}{\partial y_1}\left[D \frac{\partial^2 w}{\partial y_2^2} + D\nu \frac{\partial^2 w}{\partial y_1^2}\right]\right\}$$

(During the optimization procedures to be outlined later, the equations involving these auxiliary variables disappear and hence are not carried through the computations.)

Partial differentiation of (i.2.1) with respect to y_1 and y_2 yields the 2n system equations,

$$\begin{aligned}
&\frac{\partial x_1}{\partial y_1} = x_2 && \frac{\partial x_1}{\partial y_2} = x_3 \\
&\frac{\partial x_2}{\partial y_1} = \frac{x_4}{u} - \nu\frac{\partial x_3}{\partial y_2} && \frac{\partial x_2}{\partial y_2} = \frac{x_5}{u(1-\nu)} \\
&\frac{\partial x_3}{\partial y_1} = \frac{x_5}{u(1-\nu)} && \frac{\partial x_3}{\partial y_2} = \frac{x_6}{u} - \nu\frac{\partial x_2}{\partial y_1} \\
&\frac{\partial x_4}{\partial y_1} = x_8 - \frac{\partial x_5}{\partial y_2} && \frac{\partial x_4}{\partial y_2} = x_7
\end{aligned}$$

(i.2.3a)

(i.2.3b)

$$\begin{aligned}
&\frac{\partial x_5}{\partial y_1} = x_9 - \frac{\partial x_6}{\partial y_2} && \frac{\partial x_5}{\partial y_2} = x_8 - \frac{\partial x_4}{\partial y_1} \\
&\frac{\partial x_6}{\partial y_1} = x_{10} && \frac{\partial x_6}{\partial y_2} = x_9 - \frac{\partial x_5}{\partial y_1} \\
&\frac{\partial x_7}{\partial y_1} = u_1 && \frac{\partial x_7}{\partial y_2} = u_2 \\
&\frac{\partial x_8}{\partial y_1} = u_3 && \frac{\partial x_8}{\partial y_2} = u_4 \\
&\frac{\partial x_9}{\partial y_1} = u_5 && \frac{\partial x_9}{\partial y_2} = e^2 u^{\frac{1}{3}} x_1 - u_3 \\
&\frac{\partial x_{10}}{\partial y_1} = u_6 && \frac{\partial x_{10}}{\partial y_2} = u_7
\end{aligned}$$

These twenty first-order equations are logically equivalent to the original fourth order plate equation (i.1.1). They are now in the general form of system type II (equation (c.3.2))

$$\frac{\partial \underline{x}}{\partial y_i} = \underline{f}^i \left[\underline{y},\ \underline{x},\ \ldots\ ,\ \underline{\partial_\ell}\underline{x}\ ,\ \ldots\ ,\ \underline{u}\right]$$

where now $i = 1, 2$; $\underline{y} = (y_1, y_2)^T$; $\underline{x} = (x_1, \ldots, x_{10})^T$; and $\underline{u} = (u, u_1, \ldots, u_7)^T$.

The reason for the choice of these particular state variables (i.2.1) should be clear. Physically they may be interpreted (neglecting sign conventions) as deflection, slopes, internal bending and twisting moments, and internal shears. State variables x_7 and x_{10} have no commonly known appellation but may be given physical interpretation. Notice that as the derivative of W increases, a pyramid effect results with the introduction of additional numbers of state variables per increasing derivative order.

When differentiated with respect to the spatial coordinates y_1 and y_2, the resulting state or system equations (i.2.3a and i.2.3b respectively) also may be given a physical meaning. Equations $(\text{i.2.3a})^{1,2,3}$ and $(\text{i.2.3b})^{1,2,3}$ contain all the information regarding constitution and compatibility. For example $(\text{i. 2. 3a})^{1,2}$ and $(\text{i.2.3b})^{1}$ combined give constitution and compatibility in the y_1 direction; $(\text{i.2.3a})^{1}$ and $(\text{i.2.3b})^{1,3}$ in the y_2 direction; and $(\text{i.2.3a})^{3}$ and $(\text{i.2.3b})^{1}$ (or equivalently $(\text{i.2.3a})^{1}$ and $(\text{i.2.3b})^{2}$) the coupling y_1y_2 effect. Equations $(\text{i.2.3a})^{4,5}$ (or equivalently $(\text{i.2.3b})^{5,6}$) represent the required equilibrium relationships in the y_1 and y_2 directions. All remaining equations ensure that the end result of the decomposition process is in fact interchangeable with the original fourth order equation (i.1.1) in W. Notice, lastly, that there is a certain repetition of information between (i.2.3a) and (i.2.3b) (in the form of common equations) which appears unavoidable with type II systems.

Boundary conditions expressed in terms of the new variables are of a particularly simple form, being

$$\text{(i.2.4)} \qquad \begin{aligned} x_1\big|_{y_1 = 0,a} &= 0 \qquad & x_4\big|_{y_1 = 0,a} &= 0 \\ x_1\big|_{y_2 = 0,b} &= 0 \qquad & x_6\big|_{y_2 = 0,b} &= 0 \end{aligned}$$

(There will also be continuity conditions of state across an imaginary inner line boundary (as detailed in §H). For the moment, this requirement is put to one side. The final solution will be shown to implicitly satisfy this requirement.)

The criterion may be written

$$\text{(i.2.5)} \qquad Q = \int_{y_2}\int_{y_1} u^{\frac{1}{3}}\, dy_1 dy_2$$

§I.3 SOLUTION TO THE DESIGN ILLUSTRATION

With the state equations and criterion in hand, the Hamiltonian,

$$H = -u^{\frac{1}{3}} + \Big\{ \psi_1^1 x_2 + \psi_2^1\left[\frac{x_4}{u} - \nu\frac{\partial x_3}{\partial y_2}\right] + \psi_3^1 \frac{x_5}{u(1-\nu)} + \psi_4^1\left[x_8 - \frac{\partial x_5}{\partial y_2}\right]$$

$$+ \psi_5^1\left[x_9 - \frac{\partial x_6}{\partial y_2}\right] + \psi_6^1 x_{10} + \psi_7^1 u_1 + \psi_8^1 u_3 + \psi_9^1 u_5 + \psi_{10}^1 u_6\Big\}$$

$$\text{(i.3.1)} \qquad + \Big\{\psi_1^2 x_3 + \psi_2^2 \frac{x_5}{u(1-\nu)} + \psi_3^2\left[\frac{x_6}{u} - \nu\frac{\partial x_2}{\partial y_1}\right] + \psi_4^2 x_7$$

$$+ \psi_5^2\left[x_8 - \frac{\partial x_4}{\partial y_1}\right] + \psi_6^2\left[x_9 - \frac{\partial x_5}{\partial y_1}\right] + \psi_7^2 u_2 + \psi_8^2 u_4$$

$$+ \psi_9^2\left[e^2 u^{\frac{1}{3}} x_1 - u_3\right] + \psi_{10}^2 u_7\Big\}$$

H is maximized with respect to each control for the optimum solution. This implies a global maximization of H over the controls. Notice that H is nonlinear in u but linear in u_i , $i = 1, \ldots, 7$. This is equivalent to a singular condition (§L) for the last seven controls. (The optimality requirement in this case, where no constraints on u_i, $i = 1, \ldots, 7$, exist, is that the coefficients of u_i, denoted σ_i , $i = 1, \ldots, 7$, be maintained at zero over the domain of the plate. See §L for the basis of this statement.)

For unconstrained u, the stationary values in H for this variable are given by

$$\text{(i.3.2)} \qquad \frac{\partial H}{\partial u} = 0 = -\frac{1}{3}u^{-\frac{2}{3}} - \frac{\psi_2^1 x_4}{u^2} - \frac{\psi_3^1 x_5}{u^2(1-\nu)} - \frac{\psi_2^2 x_5}{u^2(1-\nu)}$$

$$- \frac{\psi_3^2 x_6}{u^2} + \frac{1}{3} \psi_9^2 e^2 x_1 u^{-\frac{2}{3}}$$

This is a necessary condition for a local maximum of H with respect to u.

The coefficients of u_i, i = 1, ... , 7, in (i.3.1) are

(i.3.3)

$$\sigma_1 = \frac{\partial H}{\partial u_1} = \psi_7^1 \qquad \sigma_2 = \frac{\partial H}{\partial u_2} = \psi_7^2$$

$$\sigma_3 = \frac{\partial H}{\partial u_3} = \psi_9^1 - \psi_9^2 \qquad \sigma_4 = \frac{\partial H}{\partial u_4} = \psi_8^2$$

$$\sigma_5 = \frac{\partial H}{\partial u_5} = \psi_9^1 \qquad \sigma_6 = \frac{\partial H}{\partial u_6} = \psi_{10}^2$$

$$\sigma_7 = \frac{\partial H}{\partial u_7} = \psi_{10}^2$$

It will be shown that by a suitable substitution, the requirement $\sigma_i = 0$, for all y_1, y_2, will be implicitly satisfied. In so doing much of the formal argument relating to singular conditions is circumvented.

The differential equations governing the behaviour of the adjoint variables read

$$\frac{\partial \psi_1^1}{\partial y_1} + \frac{\partial \psi_1^2}{\partial y_2} = -\psi_9^2 e^2 u^{\frac{1}{3}}$$

$$\frac{\partial \psi_2^1}{\partial y_1} + \frac{\partial \psi_2^2}{\partial y_2} = -\psi_1^1 - \nu \frac{\partial \psi_3^2}{\partial y_1}$$

$$\frac{\partial \psi_3^1}{\partial y_1} + \frac{\partial \psi_3^2}{\partial y_2} = -\nu \frac{\partial \psi_2^1}{\partial y_2} - \psi_1^2$$

$$\frac{\partial \psi_4^1}{\partial y_1} + \frac{\partial \psi_4^2}{\partial y_2} = - \frac{\psi_2^1}{u} - \frac{\partial \psi_5^2}{\partial y_1}$$

$$\text{(i.3.4)} \qquad \frac{\partial \psi_5^1}{\partial y_1} + \frac{\partial \psi_5^2}{\partial y_2} = - \frac{\psi_3^1}{u(1-\nu)} - \frac{\partial \psi_4^1}{\partial y_2} - \frac{\psi_2^2}{u(1-\nu)} - \frac{\partial \psi_6^2}{\partial y_1}$$

$$\frac{\partial \psi_6^1}{\partial y_1} + \frac{\partial \psi_6^2}{\partial y_2} = - \frac{\partial \psi_5^1}{\partial y_2} - \frac{\psi_3^2}{u}$$

$$\frac{\partial \psi_7^1}{\partial y_1} + \frac{\partial \psi_7^2}{\partial y_2} = - \psi_4^2$$

$$\frac{\partial \psi_8^1}{\partial y_1} + \frac{\partial \psi_8^2}{\partial y_2} = - \psi_4^1 - \psi_5^2$$

$$\frac{\partial \psi_9^1}{\partial y_1} + \frac{\partial \psi_9^2}{\partial y_2} = - \psi_5^1 - \psi_6^2$$

$$\frac{\partial \psi_{10}^1}{\partial y_1} + \frac{\partial \psi_{10}^2}{\partial y_2} = - \psi_6^1$$

with natural boundary conditions;
Along $y_1 = 0,a$

$$\text{(i.3.4a)} \qquad \begin{gathered} - \psi_2^1 - \nu\psi_3^2 = 0, \ - \psi_3^1 = 0, \ - \psi_5^1 - \psi_6^2 = 0, \ - \psi_6^1 = 0, \\ - \psi_7^1 = 0, \ - \psi_8^1 = 0, \ - \psi_9^1 = 0, \ - \psi_{10}^1 = 0 \end{gathered}$$

Along $y_2 = 0,b$

$$\text{(i.3.4b)} \qquad \begin{gathered} - \psi_2^2 = 0, \ -\psi_3^2 - \nu\psi_2^1 = 0, \ - \psi_4^2 = 0, \ - \psi_5^2 - \psi_4^1 = 0, \\ - \psi_7^2 = 0, \ - \psi_8^2 = 0, \ - \psi_9^2 = 0, \ - \psi_{10}^2 = 0 \end{gathered}$$

For an optimal solution to be gained of this problem, equations (i.2.3) (with (i.2.4)) and equations (i.3.4) (with (i.3.4a,b)) have to be solved. (i.2.3) and (i.3.4) are linked by the control equation (i.3.2) and the requirement that $\sigma_1, \ldots, \sigma_7$ remain zero over the optimal solution. In total there are ten state variables, twenty adjoint variables and eight control variables for which solutions have to be found. However these numbers may be reduced quite significantly as a result of the elementary nature of the equations. Many of the variables are seen to be identically zero, while some variables such as the auxiliary control variables may be eliminated at the very start of the computations.

By inspection it may be seen that

$$\psi_1^1 = \beta^2 x_8 \qquad\qquad \psi_1^2 = \beta^2 x_9$$

$$\psi_2^1 = -\beta^2 u \frac{\partial x_2}{\partial y_1} \qquad\qquad \psi_3^2 = -\beta^2 u \frac{\partial x_3}{\partial y_2}$$

$$\psi_3^1 = \psi_2^2 = -\beta^2 x_5$$

(i.3.5)

$$\psi_4^1 + \psi_5^2 = \beta^2 x_2$$

$$\psi_5^1 + \psi_6^2 = \beta^2 x_3$$

$$\psi_8^1 = \psi_9^2 = -\beta^2 x_1$$

with the remaining adjoint variables zero. β^2 is an undetermined constant and may be thought of as a modal amplitude factor on the states (but not the geometry - that is the control - as may be anticipated). From (i.3.2), β^2 has to be positive for $u^{-\frac{2}{3}}(0,0)$ to be positive; hence the choice of a constant squared. The above substitution (i.3.5) is consistent with the boundary conditions on $\underline{\psi}^1$, $\underline{\psi}^2$ and $\underline{x}$ and maintains σ_i, i=1, ... , 7 zero for all (y_1, y_2).

The control equation (i.3.2) then simplifies to become (in terms of the original plate symbols to enable a comparison with §G and §K)

(i.3.6)

$$D^{-\frac{2}{3}} = \frac{3\beta^2}{[1 + \beta^2 e^2 W^2]} \left\{ \frac{\partial^2 W}{\partial y_1^2}\left(\frac{\partial^2 W}{\partial y_1^2} + \nu \frac{\partial^2 W}{\partial y_2^2}\right) + 2(1-\nu)\left(\frac{\partial^2 W}{\partial y_1 \partial y_2}\right)^2 + \frac{\partial^2 W}{\partial y_2^2}\left(\frac{\partial^2 W}{\partial y_2^2} + \nu \frac{\partial^2 W}{\partial y_1^2}\right)\right\}$$

Equation (i.3.6) corresponds to equation (g.3.2) and hence the two system models and associated optimality conditions lead to the same solution as anticipated. β^2 in (i.3.6) is equivalent to α^2 in (g.3.2) and the choice of $\alpha^2=1$ in the numerical solution in §G is consistent with β^2 being a modal amplitude factor which only affects the states (constant multiples of β^2) and in particular W, but not the solution given for D.

For the solution found for D (=u in the present section), the Hamiltonian can be seen (by graphical means) to he concave and symmetric with respect to u. H therefore has only one stationary value over u and hence equation (i.3.2) yields the global maximum value. The solution is optimal according to the maximum principle.

I.4 AN ALTERNATIVE TREATMENT

I.4.1 Introduction. For comparison purposes (and also to highlight a special form of equation (c.3.2), as considered in the fundamental work of Lurie 1963) another reduction to a type II format is proposed and the associated solution to the illustration given.

I.4.2 Outline of the reduction. It is noticed that the system equations given in (i.2.3a,b) contain state derivative terms on the right hand side. These may be eliminated for example by modifying x_8 and x_9 and introducing two more state variables. That is, the differential equations are reduced in order but increased in number. In particular, denoting the new states by $\{\xi_j;\ j = 1, \ldots, 12\}$ then

$$\xi_j \triangleq x_j \qquad j = 1, \ldots, 7 \text{ and } 10$$

$$\xi_8 \triangleq x_8 - \frac{\partial}{\partial y_2}\left[D(1-\nu)\,\frac{\partial^2 W}{\partial y_1 \partial y_2}\right]$$

$$= \frac{\partial}{\partial y_1}\left[D\,\frac{\partial^2 W}{\partial y_1^2} + D\nu\,\frac{\partial^2 W}{\partial y_2^2}\right]$$

(i.4.1)
$$\xi_9 \triangleq x_9 - \frac{\partial}{\partial y_1}\left[D(1-\nu)\,\frac{\partial^2 W}{\partial y_1 \partial y_2}\right]$$

$$= \frac{\partial}{\partial y_2}\left[D\,\frac{\partial^2 W}{\partial y_2^2} + D\nu\,\frac{\partial^2 W}{\partial y_1^2}\right]$$

$$\xi_{11} \triangleq \frac{\partial}{\partial y_2}\left[D(1-\nu)\ \frac{\partial^2 w}{\partial y_1 \partial y_2}\right]$$

$$\xi_{12} \triangleq \frac{\partial}{\partial y_1}\left[D(1-\nu)\ \frac{\partial^2 w}{\partial y_1 \partial y_2}\right]$$

With this change in the definition of state, the auxiliary control variables correspondingly change. Denoting the new controls by μ and $\{\mu_k;\ k=1,\ \ldots\ ,\ 11\}$, then

$$\mu \triangleq u$$

$$\mu_k \triangleq u_k \quad , \qquad k = 1,\ 2,\ 6,\ 7$$

$$\mu_3 = \frac{\partial \xi_8}{\partial y_1} \quad , \qquad \mu_4 \quad \frac{\partial \xi_8}{\partial y_2}$$

(i.4.2)
$$\mu_5 = \frac{\partial \xi_9}{\partial y_1}$$

$$\mu_8 = \frac{\partial \xi_{11}}{\partial y_1} \quad , \qquad \mu_9 = \frac{\partial \xi_{11}}{\partial y_2}$$

$$\mu_{10} = \frac{\partial \xi_{12}}{\partial y_1} \quad , \qquad \mu_{11} = \frac{\partial \xi_{12}}{\partial y_2}\ (= \mu_8)$$

Partial differentiation of the states ξ_j, $j = 1,\ \ldots\ ,\ 12$ with respect to y_1 and y_2 gives the state equations

$$\frac{\partial \xi_1}{\partial y_1} = \xi_2 \qquad \frac{\partial \xi_1}{\partial y_2} = \xi_3$$

$$\frac{\partial \xi_2}{\partial y_1} = \frac{\xi_4 - \nu \xi_6}{\mu(1-\nu^2)} \qquad \frac{\partial \xi_2}{\partial y_2} = \frac{\xi_5}{\mu(1-\nu)}$$

$$\frac{\partial \xi_3}{\partial y_1} = \frac{\xi_5}{\mu(1-\nu)} \qquad \frac{\partial \xi_3}{\partial y_2} = \frac{\xi_6 - \nu \xi_4}{\mu(1-\nu^2)}$$

$$\frac{\partial \xi_4}{\partial y_1} = \xi_8 \qquad \frac{\partial \xi_4}{\partial y_2} = \xi_7$$

(i.4.3a) $$\frac{\partial \xi_5}{\partial y_1} = \xi_{12} \qquad \frac{\partial \xi_5}{\partial y_2} = \xi_{11}$$

(i.4.3b)

$$\frac{\partial \xi_6}{\partial y_1} = \xi_{10} \qquad \frac{\partial \xi_6}{\partial y_2} = \xi_9$$

$$\frac{\partial \xi_7}{\partial y_1} = \mu_1 \qquad \frac{\partial \xi_7}{\partial y_2} = \mu_2$$

$$\frac{\partial \xi_8}{\partial y_1} = \mu_3 \qquad \frac{\partial \xi_8}{\partial y_2} = \mu_4$$

$$\frac{\partial \xi_9}{\partial y_1} = \mu_5 \qquad \frac{\partial \xi_9}{\partial y_2} = e^2\mu^{\frac{1}{3}}\xi_1 - \mu_3 - \mu_8 - \mu_{11}$$

$$\frac{\partial \xi_{10}}{\partial y_1} = \mu_6 \qquad \frac{\partial \xi_{10}}{\partial y_2} = \mu_7$$

$$\frac{\partial \xi_{11}}{\partial y_1} = \mu_8 \qquad \frac{\partial \xi_{11}}{\partial y_2} = \mu_9$$

$$\frac{\partial \xi_{12}}{\partial y_1} = \mu_{10} \qquad \frac{\partial \xi_{12}}{\partial y_2} = \mu_{11}$$

which are equivalent to (i.2.3a,b) and also the original fourth order equation in W,(i.1.1). (i.4.3) are of the general form

(i.4.4) $$\frac{\partial \underline{\xi}}{\partial y_i} = \underline{f}'^{i}[\underline{y}, \underline{\xi}, \underline{\mu}] \qquad \underline{\mu} = (\mu, \mu_1, \ldots, \mu_{11})^T$$

and have right hand sides independent of state derivatives. This general form is the case originally considered by Lurie (1963). (See also Butkovskii et al 1968.) The necessary conditions for optimality are again as summarized in §H.2 but with the state derivative terms omitted.

That is, it is a special case of the present type II system.

In reducing the state equations to a form free of state derivatives on the right hand side, the intuitively pleasing characteristic of states having accepted meanings has been partly lost. In particular states ξ_j, $j = 1, \ldots, 6$ only, now have accepted meanings. The subdivision into constitutive, compatibility and equilibrium equations may be identified correspondingly to that outlined in §I.2. The resulting state equations are simpler and this in turn produces a simpler Hamiltonian and adjoint equations (although increased in number).

State boundary conditions and criterion remain the same, substituting ξ_j and μ for x_j and u respectively. See equations (i.2.4) and (i.2.5).

<u>I.4.3 Solution to the optimization problem.</u> The Hamiltonian reads

$$
\begin{aligned}
H = &- \mu^{\frac{1}{3}} + \{\lambda_1^1\xi_2 + \lambda_2^1\left(\frac{\xi_4-\nu\xi_6}{\mu(1-\nu^2)}\right) + \lambda_3^1\frac{\xi_5}{\mu(1-\nu)} + \lambda_4^1\xi_8 \\
&+ \lambda_5^1\xi_{12} + \lambda_6^1\xi_{10} + \lambda_7^1\mu_1 + \lambda_8^1\mu_3 + \lambda_9^1\mu_5 + \lambda_{10}^1\mu_6 \\
&+ \lambda_{11}^1\mu_8 + \lambda_{12}^1\mu_{10}\} + \{\lambda_1^2\xi_3 + \lambda_2^2\frac{\xi_5}{\mu(1-\nu)} + \lambda_3^2\left(\frac{\xi_6-\nu\xi_4}{\mu(1-\nu^2)}\right) \\
&+ \lambda_4^2\xi_7 + \lambda_5^2\xi_{11} + \lambda_6^2\xi_9 + \lambda_7^2\mu_2 + \lambda_8^2\mu_4 \\
&+ \lambda_9^2(e^2\mu^{\frac{1}{3}}\xi_1 - \mu_3 - \mu_8 - \mu_{11}) + \lambda_{10}^2\mu_7 + \lambda_{11}^2\mu_9 + \lambda_{12}^2\mu_{11}\}
\end{aligned}
\tag{i.4.5}
$$

For stationary μ

$$
\begin{aligned}
\frac{\partial H}{\partial \mu} = 0 = &- \frac{1}{3}\mu^{-\frac{2}{3}} - \lambda_2^1\left(\frac{\xi_4-\nu\xi_6}{\mu^2(1-\nu^2)}\right) - \lambda_3^1\frac{\xi_5}{\mu^2(1-\nu)} \\
&- \lambda_2^2\frac{\xi_5}{\mu^2(1-\nu)} - \lambda_3^2\left(\frac{\xi_6-\nu\xi_4}{\mu^2(1-\nu^2)}\right) + \frac{1}{3}\lambda_9^2 e^2\mu^{-\frac{2}{3}}\xi_1
\end{aligned}
\tag{i.4.6}
$$

The coefficients of μ_k, $k = 1, \ldots, 11$ are

(i.4.7)
$$
\begin{aligned}
\sigma_1 &= \lambda_7^1 & \sigma_7 &= \lambda_{10}^2 \\
\sigma_2 &= \lambda_7^2 & \sigma_8 &= \lambda_{11}^1 - \lambda_9^2 \\
\sigma_3 &= \lambda_8^1 - \lambda_9^2 & \sigma_9 &= \lambda_{11}^2 \\
\sigma_4 &= \lambda_8^2 & \sigma_{10} &= \lambda_{12}^1 \\
\sigma_5 &= \lambda_9^1 & \sigma_{11} &= \lambda_{12}^2 - \lambda_9^2 \\
\sigma_6 &= \lambda_{10}^1 & &
\end{aligned}
$$

The adjoint equations become

$$\frac{\partial \lambda_1^1}{\partial y_1} + \frac{\partial \lambda_1^2}{\partial y_2} = -\lambda_9^2 e^2 \mu^{\frac{1}{3}}$$

$$\frac{\partial \lambda_2^1}{\partial y_1} + \frac{\partial \lambda_2^2}{\partial y_2} = -\lambda_1^1$$

$$\frac{\partial \lambda_3^1}{\partial y_1} + \frac{\partial \lambda_3^2}{\partial y_2} = -\lambda_1^2$$

$$\frac{\partial \lambda_4^1}{\partial y_1} + \frac{\partial \lambda_4^2}{\partial y_2} = -\frac{\lambda_2^1}{\mu(1-\nu^2)} + \frac{\nu\lambda_3^2}{\mu(1-\nu^2)}$$

$$\frac{\partial \lambda_5^1}{\partial y_1} + \frac{\partial \lambda_5^2}{\partial y_2} = -\frac{\lambda_3^1}{\mu(1-\nu)} - \frac{\lambda_2^2}{\mu(1-\nu)}$$

(i.4.8)
$$\frac{\partial \lambda_6^1}{\partial y_1} + \frac{\partial \lambda_6^2}{\partial y_2} = \frac{\nu\lambda_2^1}{\mu(1-\nu^2)} - \frac{\lambda_3^2}{\mu(1-\nu^2)}$$

$$\frac{\partial \lambda_7^1}{\partial y_1} + \frac{\partial \lambda_7^2}{\partial y_2} = -\lambda_4^2$$

$$\frac{\partial\lambda_8^1}{\partial y_1} + \frac{\partial\lambda_8^2}{\partial y_2} = -\lambda_4^1$$

$$\frac{\partial\lambda_9^1}{\partial y_1} + \frac{\partial\lambda_9^2}{\partial y_2} = -\lambda_6^2$$

$$\frac{\partial\lambda_{10}^1}{\partial y_1} + \frac{\partial\lambda_{10}^2}{\partial y_2} = -\lambda_6^1$$

$$\frac{\partial\lambda_{11}^1}{\partial y_1} + \frac{\partial\lambda_{11}^2}{\partial y_2} = -\lambda_5^2$$

$$\frac{\partial\lambda_{12}^1}{\partial y_1} - \frac{\partial\lambda_{12}^2}{\partial y_2} = -\lambda_5^1$$

with natural boundary conditions,
along $y_1 = 0,a$

(i.4.8a) $$\lambda_2^1 = \lambda_3^1 = \lambda_5^1 = \ldots = \lambda_{12}^1 = 0$$

along $y_2 = 0,b$

(i.4.8b) $$\lambda_2^2 = \ldots = \lambda_5^2 = \lambda_7^2 = \ldots = \lambda_{12}^2 = 0$$

The analogous substitutions to (i.3.5) are

(i.4.9)
$$\begin{aligned}
\lambda_1^1 &= \kappa^2(\xi_8 + \xi_{11}) & \lambda_1^2 &= -\kappa^2(\xi_9 + \xi_{12}) \\
\lambda_2^1 &= \kappa^2\xi_4 & \lambda_2^2 &= -\kappa^2\xi_5 \\
\lambda_3^1 &= -\kappa^2\xi_5 & \lambda_3^2 &= -\kappa^2\xi_6 \\
\lambda_4^1 &= \kappa^2\xi_2 & \lambda_5^2 &= \kappa^2\xi_2
\end{aligned}$$

$$\lambda_5^1 = \kappa^2\xi_3 \qquad\qquad \lambda_6^2 = \kappa^2\xi_3$$

$$\lambda_8^1 = \lambda_{11}^1 = \lambda_9^2 = \lambda_{12}^2 = -\kappa^2\xi_1$$

with the remaining adjoint variables zero. κ^2 is a constant with a parallel meaning to β^2 of the previous article (§I.3). The substitutions (i.4.9) are consistent with (i.4.8a,b) and the singular requirements on σ_i, i=1, ... , 11 , and lead to equations (i.3.6) (from (i.4.6)) and (i.1.1) (from (i.4.3) or (i.4.8)) to be solved for optimality. This is the same pair of equations as evolved in §I.3 and hence leads to the same solution. Equation (i.3.6) is also equivalent to the equation obtained by Armand (Armand 1972, equation 2.10, p120) using a different approach. (See §L and §C for comments on his approach.)

I.5 SENSITIVITY DISCUSSION

I.5.1 Parameter and control sensitivity. Having derived an optimal solution it would be instructive to investigate the effect of changes in the system parameters and control on the system behaviour. The engineering significance of the solution (that is whether it is sensitive or insensitive to these changes) may then be estimated.

For the purposes of the analysis, the control and parameters have been grouped under a 'parameter vector', $\underline{p} = (p_1, p_2)^T = (e^2, u)^T$, and all further discussion in this and the following subarticle (§I.5.2) relating to parameter variations will implicitly include control variations. It is assumed that the values of e^2 and u may be varied independently. (This will be true for any given material composing the structure and reference structure.) To further facilitate the discussion, without seriously affecting the conclusions, sensitivity will only be examined over the interval $[0,a]$. The equivalent state variable and control choice corresponds to (a.2.2) (the lumped version of x_1, x_2, x_4, x_8 and u respectively of section §I.3) giving state equations

$$\begin{bmatrix} \dfrac{dx_1}{dy} \\ \\ \dfrac{dx_2}{dy} \end{bmatrix} \qquad \begin{bmatrix} x_2 \\ \\ \dfrac{x_4}{u} \end{bmatrix}$$

$$(i.5.1) \qquad \begin{bmatrix} \dfrac{dx_4}{dy} \\ \\ \dfrac{dx_8}{dy} \end{bmatrix} = \begin{bmatrix} x_8 \\ \\ e^2 u^{\frac{1}{3}} x_1 \end{bmatrix}$$

Introduce a sensitivity coefficient v defined as the change in state due to parameter variations. Formally, for a continuous or global variation;

$$(i.5.2) \qquad v_{ij}(\underline{p},y) = \frac{\partial x_i(\underline{p},y)}{\partial p_j} \qquad \begin{pmatrix} i = 1,2,4,8 \\ j = 1,2 \end{pmatrix}$$

(v_{ij} are essentially gradient functions and may be interpreted as the second terms in Taylor's series expansions of the variation in the states x_i due to small variations in the parameters $p_j \to p_j + \delta p_j$. For δp_j small,

$$x_i(\underline{p},y) \to x_i(\underline{p} + \underline{\delta p}, y)$$
$$\to x_i(\underline{p}, y) + \sum_j \frac{\partial x_i(\underline{p},y)}{\partial p_j} \delta p_j + \ldots \quad)$$

Noting that the system equation is of the general form

$$(i.5.3) \qquad \frac{d\underline{x}}{dy} = \underline{f}[\underline{x},\underline{p}]$$

for a variation in the parameters $\underline{p} \to \underline{p} + \underline{\delta p}$, there is a corresponding variation in the state $\underline{x} \to \underline{x} + \delta\underline{x}$, related through

$$\frac{d(\underline{x} + \underline{\delta x})}{dy} = \underline{f}[\underline{x} + \underline{\delta x}, \underline{p} + \underline{\delta p}]$$

Expanding the right hand side in a Taylor's series about $\underline{f}[\underline{x},\underline{p}]$

$$\frac{d(\underline{x} + \delta\underline{x})}{dy} = \underline{f}[\underline{x}\ ,\ \underline{p}] + \frac{\partial \underline{f}[\underline{x}\ ,\ \underline{p}]}{\partial \underline{x}}\,\delta\underline{x} + \frac{\partial \underline{f}[\underline{x}\ ,\ \underline{p}]}{\partial \underline{p}}\,\delta\underline{p} + \ldots$$

Using (i.5.3)

$$\delta\left(\frac{d\underline{x}}{dy}\right) = \frac{\partial \underline{f}[\underline{x}\ ,\ \underline{p}]}{\partial \underline{x}}\,\delta\underline{x} + \frac{\partial \underline{f}[\underline{x}\ ,\ \underline{p}]}{\partial \underline{p}}\,\delta\underline{p}$$

Dividing by $\delta\underline{p}$, keeping in mind that the equation applies for all $\underline{p}$ including $\underline{p} = 0$ and taking the limit as $\delta\underline{p} \to 0$, then a sensitivity equation (see for example Miller and Murray 1958, Chang 1961, Dorato 1963, Tomovic 1963, Tomovic and Vukobratovic 1972) is obtained, linear (but y-variant) in the sensitivity coefficients $\underline{v}$;

$$\text{(i.5.4)} \qquad \frac{d}{dy}\left[\frac{\partial \underline{x}}{\partial \underline{p}}\right] = \left[\frac{\partial \underline{f}}{\partial \underline{x}}\right]\left[\frac{\partial \underline{x}}{\partial \underline{p}}\right] + \left[\frac{\partial \underline{f}}{\partial \underline{p}}\right] \qquad \text{or}$$

$$\text{(i.5.4)*} \qquad \frac{d\underline{v}}{dy} = \frac{\partial \underline{f}}{\partial \underline{x}}\,\underline{v} + \frac{\partial \underline{f}}{\partial \underline{p}}$$

where $\underline{v}$ is a 4 x 2 matrix of components $v_{ij} = \dfrac{\partial x_i}{\partial p_j}$ and the parameters have been varied independently. Boundary conditions on (i.5.4)* are $v_{ij} = 0$ when $x_i = 0$ $(i = 1,2,4,8)$ at $y = 0$ and/or a. Both $\dfrac{\partial \underline{f}}{\partial \underline{x}}$ and $\dfrac{\partial \underline{f}}{\partial \underline{p}}$ matrices are evaluated about the optimal values of their arguments.

Performing the differentiations;

$$\begin{bmatrix} \dfrac{dv_{11}}{dy} & \dfrac{dv_{12}}{dy} \\[2ex] \dfrac{dv_{21}}{dy} & \dfrac{dv_{22}}{dy} \\[2ex] \dfrac{dv_{41}}{dy} & \dfrac{dv_{42}}{dy} \\[2ex] \dfrac{dv_{81}}{dy} & \dfrac{dv_{82}}{dy} \end{bmatrix} = \begin{bmatrix} & 1 & & \\[2ex] & & \dfrac{1}{u} & \\[2ex] & & & 1 \\[2ex] e^2u^{\frac{1}{3}} & & & \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\[2ex] v_{21} & v_{22} \\[2ex] v_{41} & v_{42} \\[2ex] v_{81} & v_{82} \end{bmatrix} + \begin{bmatrix} & \\[2ex] & \dfrac{-x_4}{u^2} \\[2ex] & \\[2ex] u^{\frac{1}{3}}x_1 & \frac{1}{3}e^2u^{-\frac{2}{3}}x_1 \end{bmatrix}$$

with boundary conditions

$$v_{11}\Big|_{0,a} = 0 \qquad v_{41}\Big|_{0,a} = 0$$

$$v_{12}\Big|_{0,a} = 0 \qquad v_{42}\Big|_{0,a} = 0$$

If now the optimal values for the variables are substituted, then it is a straightforward matter to solve for v_{ij} ($i = 1,2,4,8$; $j = 1,2$). Table I.5.1 gives these sensitivity coefficients for the specific numerical problem mentioned in section §G. The same value for e^2 as used in §G has been used in the calculations - here $e^2 = 1$ corresponds to an aluminium reference rod 10" x 1" x 1/10" (See Armand 1972). Units are x_1(in.), x_2(in./in.), x_4(lb in.), x_8(lb in./in.), u(lb in.), e^2(lb$^{2/3}$/in.$^{7/3}$) and y(in.). For the uniform reference member $u^{2/3} \simeq 100\, e^2$ but $v_{i1} \simeq 1000\, v_{i2}$, $i = 1,2,4,8$, and so the design is about ten times more sensitive to variations in e^2 than the optimal control. However the design is very sensitive to changes in either u or e^2.

<u>I.5.2 Sensitivity in the criterion.</u> Consider a general criterion with both state and parameter arguments (the terminology 'parameter' embraces the control - see §I.5.1). For variations in the parameters about the optimum, $\underline{\hat{p}} \to \underline{\hat{p}} + \underline{\delta p}$ which in turn changes $\underline{x} \to \underline{x} + \underline{\delta x}$ where $\underline{\delta x}$ is related to $\underline{\delta p}$ through the sensitivity coefficients $v_{ij} = \partial x_i/\partial p_j$, themselves the solution of (i.5.4)*.

The criterion $Q(\underline{x}\, ,\, \underline{\hat{p}}) \to Q(\underline{x} + \underline{\delta x}\, ,\, \underline{\hat{p}} + \underline{\delta p})$ and the change in Q using a Taylor's series expansion about Q (assuming Q possesses the necessary derivatives) becomes

$$\Delta Q = Q(\underline{x} + \underline{\delta x}\, ,\, \underline{\hat{p}} + \underline{\delta p}) - Q(\underline{x}\, ,\, \underline{\hat{p}})$$

$$= Q(\underline{x}\, ,\, \underline{\hat{p}}) + \sum_j \frac{\partial Q(\underline{x}\, ,\, \underline{\hat{p}})}{\partial \hat{p}_j}\, \delta p_j + \sum_i \frac{\partial Q(\underline{x}\, ,\, \underline{\hat{p}})}{\partial x_i}\, \delta x_i \qquad \text{(i.5.5)}$$

y	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
v_{11}	41.2	41.2	40.3	38.5	35.8	32.2	27.7	22.3	15.9	8.51	
v_{12}	-0.030	-0.030	-0.029	-0.028	-0.026	-0.024	-0.020	-0.016	-0.010	-0.006	
x_1	0.800	0.800	0.783	0.748	0.696	0.626	0.539	0.434	0.311	0.168	
v_{21}		-1.80	-3.61	-5.41	-7.21	-9.03	-10.9	-12.7	-14.7	-17.0	-19.8
v_{22}		0.0013	0.0026	0.0040	0.0053	0.0066	0.0080	0.0094	0.011	0.013	0.015
x_2		-0.035	-0.070	-0.105	-0.140	-0.175	-0.210	-0.247	-0.286	-0.331	-0.393
v_{41}	-4357.2	-4357.2	-4245.3	-4021.5	-3689.0	-3254.9	-2728.9	-2124.0	-1456.0	-741.99	
v_{42}	3.113	3.113	3.033	2.873	2.636	2.326	1.950	1.518	1.040	0.530	
x_4	-83.9	-83.9	-81.8	-77.5	-71.2	-62.9	-52.9	-41.4	-28.6	-15.0	
v_{81}		223.79	447.58	664.58	868.11	1051.7	1209.5	1336.2	1428.1	1483.8	1505.8
v_{82}		-0.160	-0.320	-0.475	-0.620	-0.751	-0.865	-0.955	-1.021	-1.060	-1.075
x_8		4.26	8.52	12.6	16.5	20.0	23.0	25.5	27.2	28.3	28.7
u	1208.6	1208.6	1193.0	1145.9	1066.1	952.6	805.8	628.5	429.5	228.1	

Table I.5.1 Sensitivity Coefficients.

$$+ \sum_{j,k} \frac{\partial^2 Q(\underline{x}, \hat{\underline{p}})}{\partial \hat{p}_j \partial \hat{p}_k} \frac{\delta p_j \delta p_k}{2!} + \sum_{h,i} \frac{\partial^2 Q(\underline{x}, \hat{\underline{p}})}{\partial x_h \partial x_i} \frac{\delta x_h \delta x_i}{2!}$$

$$+ \ldots - Q(\underline{x}, \hat{\underline{p}})$$

For the problem at hand $Q = \int_0^a u^{\frac{1}{3}} dy$. The first derivatives about the optimal solution by definition vanish, and hence the change in the criterion to terms of order $(\delta u)^3$ is given by the second derivative terms (positive for a minimum) (see Rohrer and Sobral 1965 for related reasoning). That is

$$\Delta Q = \frac{\partial^2 Q}{\partial u^2} \frac{(\delta u)^2}{2} = - \frac{1}{9} \int_0^a u^{-\frac{5}{3}} dy (\delta u)^2 \simeq - 2.68 \times 10^{-5} \; (\delta u)^2 \tag{i.5.6}$$

and

$$\frac{\Delta Q}{Q} \simeq - 2.72 \times 10^{-6} (\delta u)^2$$

From the direct proportionality of ΔQ to $(\delta u)^2$ in (i.5.6), the criterion will be more sensitive where the control change is greatest, that is approaching the boundaries. Q is found to be approximately thirty times more sensitive close to the boundary than at the centreline. However the overall sensitivity of Q to changes in the control is very small.

I.6 DISCUSSION

From the symmetry of the problem it might be anticipated that the system model type II would be a desirable representation to adopt. However any advantage in using this model is not readily apparent. The designer's preference in this case would dictate the use of one system model in preference to another.

The present scheme breeds a very large number of equations (often with common information), though of simple construction. Moreover there does not appear to be any way of avoiding the introduction of the auxiliary equation control variables. The introduction of these variables is not appealing. System model types I and III avoid their introduction (though it is possible to incorporate them in III - see §K.4) but at the same time I neglects the true two dimensional behaviour of the structure while III does not appear to contain enough information. Justification for the use of type I comes in problems where the behaviour is distinctly different in two or more of the independent variable directions. Both models I and II (but not III) have the lumped system model as a special case, and no distinction can be made on this ground.

The route to the optimal solution again involves the solution of a set of auxiliary (adjoint) equations simultaneous to the system equations. Natural boundary conditions on the adjoint variables also assume a similar form as in the previous illustration. The maximum principle (here in a distributed parameter form) thus offers considerable potential for the systematic solution of optimal control problems. There was however a certain awkwardness in the manipulation of the necessary conditions, which is inherent in any treatment of partial differential equations with split boundary conditions. This awkwardness would become more apparent for more complicated design problems and would inevitably necessitate the use of numerical solution techniques. Irrespective of the final numerical solution process adopted, the necessary conditions give an ideal general description of the solution; in keeping with other analytical methods the solution is applicable for a whole class of problems.

In any design, sensitivity considerations are fundamental to an understanding of the design and should be taken into account; their obvious usage allows for discrepancies between the mathematical model and the physical system. Optimum systems insensitive to variations about the optimum are to be preferred. For the problem at hand, the optimal solution was found to be very sensitive to parameter variations while the criterion was relatively insensitive to variations in its argument. The sensitivity computations performed were illustrative of the routine required in estimating sensitivity and completeness of treatment is not suggested; for example, only variations were considered that did not change the system description (including boundary conditions). However the computations emphasize that sensitivity may be considered an integral part of the design procedure. It is hoped that this illustration has focused attention on the sensitivity problem in structural design.

§J DERIVATION OF NECESSARY CONDITIONS FOR OPTIMALITY: SYSTEM TYPE III

J.1 INTRODUCTION

J.1.1 Outline. This section gives the derivation of the necessary conditions for optimality for the third system type (III). Conventional calculus of variations arguments are used, namely setting the first variation of the criterion to zero and allowing free variations of the state and control, to obtain the control that yields the stationary value of the criterion. For this control to be also minimizing the first variation of the criterion has to be positive which leads on to a version of the maximum principle; that is some scalar quantity H, the Hamiltonian, has to be maximized over the admissible controls for optimality.

The classical calculus of variations arguments used in the derivation are the third approach to deriving necessary conditions in this thesis. Previous sections (§F and §H) use variational arguments after Rozonoer (localized variations) and dynamic programming techniques after Bellman. The three approaches can thus be shown to lead to similar results - maximum principles - providing a connection between them. Assumptions however vary between the approaches. It is remarked that the derivation arguments are not exclusive to the system type on which they are used but rather are interchangeable. Many of the references given illustrate this on system types less general than considered here.

The necessary conditions are summarized following the derivation and their usage is illustrated in the following section (§K) on the plate problem of Armand treated in several ways in previous sections (§G and §I).

Initially a generalized problem of Bolza (with side constraints and in Lagrangian and Hamiltonian notation) is solved to obtain the necessary conditions for a stationary value (Euler-Lagrange equations with natural boundary conditions). Constraints are included in the formulation to derive a special version of the maximum principle for type III systems. The treatment will closely follow the work of Lurie (1963) on a particular type II system. To correlate the results with those outlined in §F and §H, the very general conclusions derived here will be specialized so as to agree in form with the range of application treated in those two sections. The specialized form will be shown to assume a very neat statement of the necessary conditions for optimality.

Constraints in the problem formulation are treated as in most variational calculus texts, with the use of Lagrange multipliers. The technique of Lagrange multipliers adjoins the constraint to the functional being minimized and removes the necessity of prior elimination of free parameters (whether this is feasible or infeasible in particular problems). The use of Lagrange multipliers allows the problem to be treated as if it were unconstrained.

The stationary conditions obtained in the calculus of variations are based on the assumption that variations in the state and control are completely arbitrary and lead to a special case of Pontryagin's maximum principle (namely a stationary condition on the Hamiltonian with respect to the control as given by setting $\partial H/\partial \underline{u} = 0$). This however is only possible when the admissible state and control sets are unbounded. Where the admissible control set is bounded the variations in the control cannot be completely arbitrary and the usual approach of the calculus of variations no longer holds. (The same situation for controls arises in the presence of constraints on the state space where the control must be chosen without violating these constraints.)

This is an inherent restriction of the calculus of variations but may be overcome with extensions due to Weierstrass; in particular the Weierstrass-Erdmann corner conditions which give the requirements at discontinuities in extremals and extend the admissible class of controls to include piecewise continuous functions (and hence allow constraints on the controls). It is this extended form of the calculus of variations that will be followed in the following derivation.

In general the optimal solution will contain portions both on and off the constraints at a finite number of points ('corners'). Corner points may also arise in an unconstrained problem where a discontinuous control is a valid solution to the Euler-Lagrange equations. Corner points impose special requirements on the optimal solution. These requirements are contained in the Weierstrass-Erdmann corner conditions.

J.1.2 Background. Historically systems modelled according to a type III format were the first distributed parameter systems described by a set of partial differential equations for which a maximum principle was obtained. Their introduction was the beginning of the transfer from integral equation systems as pioneered by Butkovskii and Lerner to the more general differential equation systems.

A.I. Egorov's initial investigations with a type III form were on quasilinear partial differential equations (1963), proving sufficiency of the optimization for the linear case. This was generalised to sets of equations of the second order (1964) and special conditions were obtained for hyperbolic, parabolic and elliptic equations (1966, 1967a, 1967b). In all cases the basic mode of derivation of the necessary conditions followed Rozonoer's method (1959). The results for the particular sets of second order systems considered are thus stronger than those presented in this section, implying a global maximization of the Hamiltonian over the admissible controls (compared with a local maximization given here). However the results only allow first order derivatives of state on the right hand sides and are generally only initial value problems; the present section removes these restrictions. For a summary of A.I. Egorov's work, see Butkovskii (1969) where results for special controls are given.

The use of classical calculus of variations philosophy (that is the assumption of free variations) in optimal control, because of their essentially equivalent problem constructions, was early. Lurie (1963) solves the Mayer-Bolza problem for multiple integrals with special forms of type II partial differential equations as side constraints. The necessary stationary value conditions and the necessary Legendre and Weierstrass conditions (with the relationship to the maximum principle) are given for two independent variables. See also Armand (1971, 1972). Jackson (1966) treats the same variational problem and derives special results for the case where the system equations are hyperbolic and particular boundary conditions apply. Kim and Gajwani (1968), for a system type I over two independent variables and an integral criterion over the time domain only, use the methods of the calculus of variations to derive the canonical equations as necessary conditions.

J.1.3 Formulation of the problem. Consider the general problem similar to that formulated and solved for a special case of type II systems by Lurie (1963). (The results will be later specialized to coincide in form with those derived in previous sections.) The symbol $\underline{y}$ will be used to denote a two dimensional coordinate vector $\underline{y} = (y_1, y_2)^T$ belonging to a region Y in the y_1y_2 - plane with piecewise continuous boundary curves ∂Y^a and ∂Y^b (Figure J.1.1).

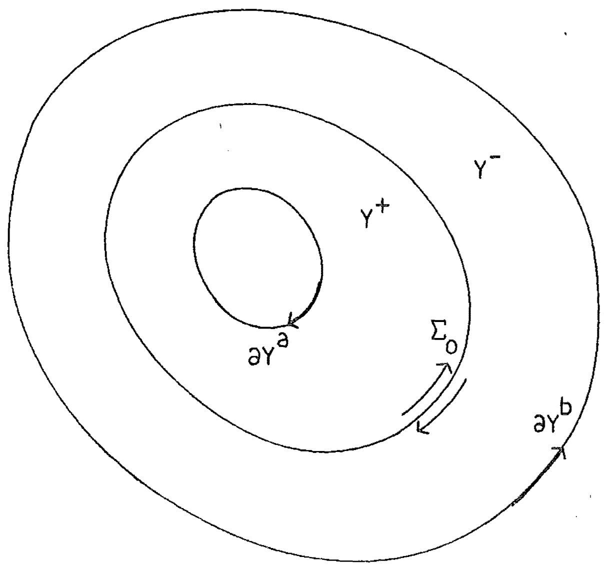

Figure J.1.1

The symbol σ will be taken as arc length of a curve, n its outward normal.

For a state vector $\underline{x}(y_1, y_2) = (x_1, \ldots, x_n)^T$ and control vector $\underline{u}(y_1, y_2) = (u_1, \ldots, u_r)^T$ defined for all $\underline{y} \in Y$, consider a system in the region Y described by the set of partial differential equations of type III form (§C.3),

$$\text{(j.1.1)} \qquad \frac{\partial^2 \underline{x}}{\partial y_1 \partial y_2} = \underline{f}[\underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}, y_1, y_2]$$

where $\ell = \ell_1$ or ℓ_2 but never (ℓ_1, ℓ_2) together; $\partial_\ell \underline{x}$ is as defined in the 'notation'. (This is a special case where $L = \ell_1$ or ℓ_2 directly.) $\underline{f} = (f_1, \ldots, f_n)^T$ is in general a nonlinear vector function of the arguments shown.

Boundary conditions on $\underline{x}$ will be given of the form

(j.1.1a) $\qquad x_i(\sigma), \ldots, \partial_{\ell-1} x_i(\sigma)$

implying lower order derivatives to the state derivative terms appearing on the right hand side of (j.1.1). These will be distributed between ∂Y^a and ∂Y^b and even over portions of ∂Y^a and ∂Y^b. That is split boundary conditions are implied. The values taken by i in (j.1.1a) are determined by the conditions of any given problem.

Consider also, associated control constraints (§D.2); the m_1 equalities

(j.1.2a) $\qquad h_k(\underline{u}, y_1, y_2) = 0 \qquad\qquad k = 1, \ldots, m_1$

and $m-m_1$ inequalities

(j.1.2b) $\qquad h_k(\underline{u}, y_1, y_2) \geq 0 \qquad\qquad k = m_1+1, \ldots, m$

where $m \leq r$.

The controls will be taken to be piecewise continuous functions of the coordinates y_1 and y_2; it is assumed that discontinuities may occur along some isolated closed smooth lines Σ_0 which may be reduced to the boundary curves ∂Y^a or ∂Y^b by continuous deformations. Specifically, for ease in computations, the region Y will be assumed to have only one line of discontinuity of the controls. Superscripts plus and minus denote alternate sides of Σ_0, determined in some consistent directional sense. The state will be assumed continuous throughout Y. Any control $\underline{u}$ which satisfies the requirements (j.1.2) and is piecewise continuous will be termed admissible.

The problem considered here is one of finding the control $\underline{u}(y_1, y_2)$ which minimizes the criterion (§D.3)

$$Q = \iint\limits_{Y} G(\underline{x}, \ \ldots \ , \partial_\ell \underline{x}, \ \ldots \ , \underline{u}, y_1, y_2) dy_1 dy_2$$

(j.1.3)

$$+ \oint\limits_{\partial Y^a} g_a(\underline{x},\sigma)d\sigma + \oint\limits_{\partial Y^b} g_b(\underline{x},\sigma)d\sigma$$

where G, g_a and g_b are scalar functions of their given arguments. This is a generalised Bolza problem with partial differential equation and control constraints. The functions G, g_a, g_b and $\underline{f}$ are assumed differentiable with respect to their arguments.

For the transformations involved in converting criteria of other forms than (j.1.3) to the form of (j.1.3), see A.I. Egorov (1964) or Butkovskii (1969). With these transformations, the above problem statement can be seen to cover a broad range of problems.

J.2 DERIVATION OF THE CONDITIONS.

J.2.1 Necessary conditions for a stationary value. The inequalities of (j.1.2b) may be removed so as to permit open variations of the controls, by introducing supplementary artificial controls $\underline{u}^* = (u^*_{m_1+1} \ , \ \ldots \ , \ u^*_m)^T$ according to

(j.1.2b)' $\quad h^*_k \triangleq h_k(\underline{u}, y_1, y_2) - (u^*_k)^2 = 0 \qquad k = m_1+1, \ \ldots \ m$

which are now in an equality constraint form and replace (j.1.2b). (See Valentine 1937, Miele 1962b.)

Also set $\underline{F} \triangleq \dfrac{\partial^2 \underline{x}}{\partial y_1 \partial y_2} - \underline{f}[\underline{x}, \ \ldots \ , \partial_\ell \underline{x}, \ \ldots \ , \underline{u}, y_1, y_2]$

from (j.1.1).

The system equations (j.1.1) are treated as differential equality constraints and together with constraints (j.1.2a) and (j.1.2b)' are adjoined to the criterion (j.1.3) by means of Lagrange multipliers $\underline{\eta}(y_1, y_2) = (\eta_1, \ \ldots \ , \eta_n)^T$, $\xi(y_1, y_2) = (\xi_1, \ \ldots \ , \xi_{m_1})^T$ and

$\xi^*(y_1, y_2) = (\xi^*_{m_1+1}, \ldots, \xi^*_m)^T$. The composite functional I is written as the sum of two portions corresponding to the regions Y^+ and Y^-.

$$I = Q + \iint_{Y^+} \{\underline{\eta}^T\underline{F} + \underline{\xi}^T\underline{h} + \xi^{*T}\underline{h}^*\}^+ dy_1 dy_2 \tag{j.2.1}$$
$$+ \iint_{Y^-} \{\underline{\eta}^T\underline{F} + \underline{\xi}^T\underline{h} + \xi^{*T}\underline{h}^*\}^- dy_1 dy_2$$

The terms in braces are always zero and hence I always equals Q, implying that I and Q are simultaneously stationary.

Introduce the Lagrangian function

$$L = G + \underline{\eta}^T\underline{F} + \underline{\xi}^T\underline{h} + \xi^{*T}\underline{h}^* \tag{j.2.2}$$

For variations in the control vector and consequently in the state vector about the optimal control and optimal state, the first variation in I may be found from (j.2.1).(Note the Lagrange multipliers do not vary.)

$$\delta I = \iint_{Y^\pm} \left\{ (\underline{\delta x})^T \frac{\partial L}{\partial \underline{x}} + (\underline{\delta u})^T \frac{\partial L}{\partial \underline{u}} + (\underline{\delta u}^*)^T \frac{\partial L}{\partial \underline{u}^*} \right.$$
$$\left. + [\delta(\partial_\ell \underline{x})]^T \frac{\partial L}{\partial[\partial_\ell \underline{x}]} + \left[\delta\left(\frac{\partial^2 \underline{x}}{\partial y_1 \partial y_2}\right)\right]^T \frac{\partial L}{\partial\left[\frac{\partial^2 \underline{x}}{\partial y_1 \partial y_2}\right]} \right\}^\pm dy_1 dy_2 \tag{j.2.3}$$
$$+ \oint_{\partial Y^a} \left\{(\underline{\delta x})^T \frac{\partial g_a}{\partial \underline{x}}\right\}^+ d\sigma + \oint_{\partial Y^b} \left\{(\underline{\delta x})^T \frac{\partial g_b}{\partial \underline{x}}\right\}^- d\sigma$$
$$+ \oint_{\Sigma_0} \left\{L\delta n\right\}^\pm d\sigma$$

where δn is the variation in the external normal of the curve Σ_o (location free to vary).

Two terms in (j.2.3) may be rewritten;

$$\iint_{Y^\pm} \left\{ \left[\delta \left(\frac{\partial^2 \underline{x}}{\partial y_1 \partial y_2} \right) \right]^T \frac{\partial L}{\partial \left[\frac{\partial^2 \underline{x}}{\partial y_1 \partial y_2} \right]} \right\}^\pm dy_1 dy_2$$

$$= \iint_{Y^\pm} \left\{ \underline{\eta}^T \frac{\partial^2 (\delta \underline{x})}{\partial y_1 \partial y_2} \right\}^\pm dy_1 dy_2$$

$$= \iint_{Y^\pm} \left\{ \frac{\partial^2}{\partial y_1 \partial y_2} (\underline{\eta}^T \delta \underline{x}) - \frac{\partial}{\partial y_1} \left(\frac{\partial \underline{\eta}^T}{\partial y_2} \delta \underline{x} \right) - \frac{\partial}{\partial y_2} \left(\frac{\partial \underline{\eta}^T}{\partial y_1} \delta \underline{x} \right) + \frac{\partial^2 \underline{\eta}^T}{\partial y_1 \partial y_2} \delta \underline{x} \right\}^\pm dy_1 dy_2$$

Using Green's theorem

$$= [\underline{\eta}^T \delta \underline{x}]_{Y^\pm} - \oint_{\partial Y^a} \delta \underline{x}^+ A^+ d\sigma - \oint_{\Sigma_o} \delta \underline{x}^\pm A^\pm d\sigma - \oint_{\partial Y^b} \delta \underline{x}^- A^- d\sigma$$

(j.2.4)

$$+ \iint_{Y^\pm} \left\{ \frac{\partial^2 \underline{\eta}^T}{\partial y_1 \partial y_2} \delta \underline{x} \right\}^\pm dy_1 dy_2$$

where $A = \left(\frac{\partial \underline{\eta}^T}{\partial y_2} \frac{dy_2}{d\sigma} - \frac{\partial \underline{\eta}^T}{\partial y_1} \frac{dy_1}{d\sigma} \right)$ and $[E]_{Y^\pm}$ denotes the increase in E over the regions $Y^\pm$.

Also

$$\iint_{Y^\pm} \left\{ \left[\delta (\partial_\ell \underline{x}) \right]^T \frac{\partial L}{\partial [\partial_\ell \underline{x}]} \right\}^\pm dy_1 dy_2$$

$$= \iint_{Y^\pm} \left\{ (\partial_\ell \delta \underline{x})^T \frac{\partial L}{\partial [\partial_\ell \underline{x}]} \right\}^\pm dy_1 dy_2$$

$$= \iint_{Y^{\pm}} \left\{ \partial_{\ell}\left\{(\partial_{\ell-1}\underline{\delta x})\left[\frac{\partial L}{\partial[\partial_{\ell}\underline{x}]}\right]\right\} - \partial_{\ell}\left\{(\partial_{\ell-2}\underline{\delta x})\partial_1\left[\frac{\partial L}{\partial[\partial_{\ell}\underline{x}]}\right]\right\}\right.$$

$$+ \partial_{\ell}\left\{(\partial_{\ell-3}\underline{\delta x})\partial_2\left[\frac{\partial L}{\partial[\partial_{\ell}\underline{x}]}\right]\right\} - \ldots + (-1)^{\ell-1}\partial_{\ell}\left\{(\underline{\delta x})\partial_{\ell-1}\left[\frac{\partial L}{\partial[\partial_{\ell}\underline{x}]}\right]\right\}$$

$$\left. + (-1)^{\ell}(\underline{\delta x})^{T}\partial_{\ell}\left[\frac{\partial L}{\partial[\partial_{\ell}\underline{x}]}\right]\right\}^{\pm} dy_1 dy_2$$

All terms except the last are now integrated. For the terms preceding the ellipsis dots, some vanish upon invoking (j.1.1a) while the optimal solution will be required to be of a form such that the remainder also vanish. Green's theorem is applied to the second last term and the whole last line becomes,

$$= \oint_{\partial Y^a} \underline{\delta x}^{+} B^{+} d\sigma + \oint_{\Sigma_0} \underline{\delta x}^{\pm} B^{\pm} d\sigma + \oint_{\partial Y^b} \underline{\delta x}^{-} B^{-} d\sigma$$

(j.2.5)

$$+ \iint_{Y^{\pm}} \left\{(-1)^{\ell}(\underline{\delta x})^{T}\partial_{\ell}\left[\frac{\partial L}{\partial[\partial_{\ell}\underline{x}]}\right]\right\}^{\pm} dy_1 dy_2$$

where $B = (-1)^{\ell_1-1}\partial_{\ell_1-1}\left[\frac{\partial L}{\partial[\partial_{\ell_1}\underline{x}]}\right]\frac{dy_2}{d\sigma} - (-1)^{\ell_2-1}\partial_{\ell_2-1}\left[\frac{\partial L}{\partial[\partial_{\ell_2}\underline{x}]}\right]\frac{dy_1}{d\sigma}$

With these substitutions (j.2.4 and j.2.5), (j.2.3) becomes

$$\delta I = \iint_{Y^{\pm}} \left\{(\underline{\delta x})^{T}\frac{\partial L}{\partial \underline{x}} + (\underline{\delta u})^{T}\frac{\partial L}{\partial \underline{u}} + (\underline{\delta u^*})^{T}\frac{\partial L}{\partial \underline{u}^*}\right.$$

$$+ (-1)^{\ell}(\underline{\delta x})^T\partial_\ell\left[\frac{\partial L}{\partial[\partial_\ell \underline{x}]}\right] + (\underline{\delta x})^T \left.\frac{\partial^2 \underline{\eta}}{\partial y_1 \partial y_2}\right\}^{\pm} dy_1 dy_2$$

(j.2.6)
$$+ \oint_{\partial Y^a} (\underline{\delta x}^+)^T\left\{\frac{\partial g_a}{\partial \underline{x}} + B\text{-}A\right\}^+ d\sigma + \oint_{\partial Y^b} (\underline{\delta x}^-)^T\left\{\frac{\partial g_b}{\partial \underline{x}} + B\text{-}A\right\}^- d\sigma$$

$$+ \int_{\Sigma_o} \left\{L\delta n + (\underline{\delta x})^T(B\text{-}A)\right\}^{\pm} d\sigma + [\underline{\eta}^T\underline{\delta x}]_{Y^\pm}$$

It is noted that the total variation of a function $f(n)$ is $\Delta f(n) = \delta f(n) + \frac{\partial f}{\partial n}\delta n$, then the line integral on Σ_o reduces to

(j.2.6a)
$$\oint_{\Sigma_o} \left\{\underline{\Delta x}^T(B\text{-}A) + \delta n\left[L - \frac{\partial \underline{x}^T}{\partial n}(B\text{-}A)\right]\right\}^{\pm} d\sigma$$

The necessary condition for a stationary value of I (equivalently Q) is that the first variation of I vanish for arbitrary $\underline{\delta x}$ and $\underline{\delta u}$ in the regions $Y^{\pm}$. Thus it is required that;

In regions $Y^{\pm}$,

(j.2.7a)
$$\left.\frac{\partial L}{\partial \underline{x}} + (-1)^{\ell}\partial_\ell\left[\frac{\partial L}{\partial[\partial_\ell \underline{x}]}\right] + \frac{\partial^2 \underline{\eta}}{\partial y_1 \partial y_2}\right|^{\pm} = 0\ ,$$

(j.2.7b)
$$\left.\frac{\partial L}{\partial \underline{u}}\right|^{\pm} = 0\ , \qquad \left.\frac{\partial L}{\partial \underline{u}^*}\right|^{\pm} = 0\ , \qquad [\underline{\eta}^T\underline{\delta x}]_{Y^\pm} = 0$$

Along boundary ∂Y^a,

(j.2.8)
$$\underline{x} \text{ given or } \left.\frac{\partial g_a}{\partial \underline{x}} + B\text{-}A\right|^{+} = 0$$

Along boundary ∂Y^b,

(j.2.9)
$$\underline{x} \text{ given or } \left.\frac{\partial g_b}{\partial \underline{x}} + B\text{-}A\right|^{-} = 0$$

Along the discontinuity curve Σ_0

$$\text{(j.2.10)}\qquad \left. B-A \right|^{\pm} = 0 \qquad \left. L - \frac{\partial \underline{x}^T}{\partial n}(B-A) \right|^{\pm} = 0$$

Introduce a scalar function, the Hamiltonian, in an analogous manner to the previous derivations,

$$\text{(j.2.11)}\qquad H = \underline{\eta}^T \underline{f} - L = \left[\frac{\partial^2 \underline{x}}{\partial y_1 \partial y_2}\right]^T \frac{\partial L}{\partial\left[\dfrac{\partial^2 \underline{x}}{\partial y_1 \partial y_2}\right]} - L$$

so that (j.1.1), (j.2.7), (j.2.8) and (j.2.9) become

$$\text{(j.1.1)}^{\circ}\qquad \frac{\partial^2 \underline{x}}{\partial y_1 \partial y_2} = \frac{\partial H}{\partial \underline{\eta}}$$

$$\text{(j.2.7a)}^{\circ}\qquad \frac{\partial^2 \underline{\eta}}{\partial y_1 \partial y_2} = \frac{\partial H}{\partial \underline{x}} + (-1)^{\ell}\partial_{\ell}\left[\frac{\partial H}{\partial[\partial_{\ell}\underline{x}]}\right]$$

$$\text{(j.2.7b,c,d)}^{\circ}\qquad \frac{\partial H}{\partial \underline{u}} = 0, \qquad \frac{\partial H}{\partial \underline{u}^*} = (2\underline{\xi}^{*T}\underline{u}^*) = 0, \qquad [\underline{\eta}^T \underline{\delta x}]_{Y^{\pm}} = 0$$

$$\text{(j.2.8)}^{\circ}\qquad \frac{\partial g_a}{\partial \underline{x}} - \left((-1)^{\ell_1 - 1}\partial_{\ell_1 - 1}\left[\frac{\partial H}{\partial[\partial_{\ell_1}\underline{x}]}\right]\frac{dy_2}{d\sigma} - (-1)^{\ell_2 - 1}\partial_{\ell_2 - 1}\left[\frac{\partial H}{\partial[\partial_{\ell_2}\underline{x}]}\right]\frac{dy_1}{d\sigma}\right)$$

$$\left. - \left(\frac{\partial \underline{\eta}}{\partial y_2}\frac{dy_2}{d\sigma} - \frac{\partial \underline{\eta}}{\partial y_1}\frac{dy_1}{d\sigma}\right)\right|^{+} = 0$$

$$\frac{\partial g_b}{\partial \underline{x}} - \left((-1)^{\ell_1 - 1}\partial_{\ell_1 - 1}\left[\frac{\partial H}{\partial[\partial_{\ell_1}\underline{x}]}\right]\frac{dy_2}{d\sigma} - (-1)^{\ell_2 - 1}\partial_{\ell_2 - 1}\left[\frac{\partial H}{\partial[\partial_{\ell_2}\underline{x}]}\right]\frac{dy_1}{d\sigma}\right)$$

$$(j.2.9)^{o} \qquad -\left(\frac{\partial \underline{\eta}}{\partial y_2}\frac{dy_2}{d\sigma} - \frac{\partial \underline{\eta}}{\partial y_1}\frac{dy_1}{d\sigma}\right)\Bigg|^{-} = 0$$

J.2.2 Summary of the necessary conditions for a stationary value.

For a region Y in the y_1y_2-plane with piecewise continuous boundaries ∂Y^a and ∂Y^b, consider a system characterized by a type III form, namely

$$(j.1.1) \qquad \frac{\partial^2 \underline{x}}{\partial y_1 \partial y_2} = \underline{f}[\underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}, y_1, y_2]$$

where $\underline{x} = (x_1, \ldots, x_n)^T$ and $\underline{u} = (u_1, \ldots, u_r)^T$ denote the state and control respectively, the latter being constrained to lie within some permissible region U defined by

$$(j.1.2a) \qquad h_k(\underline{u}, y_1, y_2) = 0 \qquad\qquad k = 1, \ldots, m_1$$

$$(j.1.2b) \qquad h_k(\underline{u}, y_1, y_2) \geq 0 \qquad\qquad k = m_1+1, \ldots, m$$

Boundary conditions on (j.1.1) will be of the form

$$(j.1.1a) \qquad x_i(\sigma), \ldots, \partial_{\ell-1} x_i(\sigma) \qquad \text{given on } \partial Y^a \text{ and } \partial Y^b$$

The generalised Bolza problem is then one of determining the state $\underline{x}$ and control $\underline{u}$ which minimize the criterion

$$(j.1.3) \qquad Q = \iint_Y G(\underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}, y_1, y_2)\, dy_1 dy_2 + \oint_{\partial Y^a} g_a(\underline{x}, \sigma)\, d\sigma + \oint_{\partial Y^b} g_b(\underline{x}, \sigma)\, d\sigma$$

subject to the conditions (j.1.2a,b) and system characteristics (j.1.1) (with j.1.1a).

The solution introduces vector Lagrange multiplier functions $\underline{\eta}$, $\underline{\xi}$, $\underline{\xi}^*$. $\eta(y_1, y_2) = (\eta_1, \ldots, \eta_n)^T$ is defined by

$$\text{(j.2.7a,d)}^{\circ} \quad \frac{\partial^2 \underline{\eta}}{\partial y_1 \partial y_2} = \frac{\partial H}{\partial \underline{x}} + (-1)^{\ell} \partial_{\ell} \left[\frac{\partial H}{\partial [\partial_{\ell} \underline{x}]} \right] \quad , \quad [\underline{\eta}^T \underline{\delta x}]_Y = 0$$

over region Y with boundary conditions
(a) along boundary ∂Y^a, $\underline{x}$ given or

$$\text{(j.2.8)}^{\circ} \quad \frac{\partial g_a}{\partial \underline{x}} - \left((-1)^{\ell_1 - 1} \partial_{\ell_1 - 1} \left[\frac{\partial H}{\partial [\partial_{\ell_1} \underline{x}]} \right] \frac{dy_2}{d\sigma} - (-1)^{\ell_2 - 1} \partial_{\ell_2 - 1} \left[\frac{\partial H}{\partial [\partial_{\ell_2} \underline{x}]} \right] \frac{dy_1}{d\sigma} \right) - \left(\frac{\partial \underline{\eta}}{\partial y_2} \frac{dy_2}{d\sigma} - \frac{\partial \underline{\eta}}{\partial y_1} \frac{dy_1}{d\sigma} \right) = 0$$

(b) along boundary ∂Y^b, $\underline{x}$ given or

$$\text{(j.2.9)}^{\circ} \quad \frac{\partial g_b}{\partial \underline{x}} - \left((-1)^{\ell_1 - 1} \partial_{\ell_1 - 1} \left[\frac{\partial H}{\partial [\partial_{\ell_1} \underline{x}]} \right] \frac{dy_2}{d\sigma} - (-1)^{\ell_2 - 1} \partial_{\ell_2 - 1} \left[\frac{\partial H}{\partial [\partial_{\ell_2} \underline{x}]} \right] \frac{dy_1}{d\sigma} \right) - \left(\frac{\partial \underline{\eta}}{\partial y_2} \frac{dy_2}{d\sigma} - \frac{\partial \underline{\eta}}{\partial y_1} \frac{dy_1}{d\sigma} \right) = 0$$

where the Hamiltonian H is defined by

$$\text{(j.2.11)} \quad \begin{aligned} & H[\underline{x}, \ldots, \partial_{\ell} \underline{x}, \ldots, \underline{\eta}, \underline{u}, y_1, y_2] \\ &= \underline{\eta}(y_1, y_2)^T \underline{f}[\underline{x}, \ldots, \partial_{\ell} \underline{x}, \ldots, \underline{u}, y_1, y_2] \\ &\quad - G[\underline{x}, \ldots, \partial_{\ell} \underline{x}, \ldots, \underline{u}, y_1, y_2] \\ &\quad - \underline{\xi}(y_1, y_2)^T \underline{h}[\underline{u}, y_1, y_2] - \underline{\xi}^*(y_1, y_2)^T \underline{h}^*[\underline{u}, y_1, y_2] \end{aligned}$$

$\underline{h}^*$ has been introduced according to (j.1.2b)'.

The control is chosen so that

$$\text{(j.2.7b,c)}^{\circ} \quad \frac{\partial H}{\partial \underline{u}} = 0 \quad , \quad \frac{\partial H}{\partial \underline{u}^*} = 2\xi^{*T}\underline{u}^* = 0$$

These two equations, together with (j.1.2a) and (j.1.2b)', give $r + (m - m_1) + m$ equations to solve for $\underline{u}$, $\underline{u}^*$, $\underline{\xi}$ and $\underline{\xi}^*$.

The continuity conditions for curves of discontinuities in the controls are given by the relations (j.2.10).

Equations (j.2.7)$^{\circ}$ are the equivalent of Euler-Lagrange equations and conditions (j.2.8)$^{\circ}$ and (j.2.9)$^{\circ}$ are the related transversality conditions (reducing to 'natural' boundary conditions in special cases) of the calculus of variations. They are necessary conditions for a stationary value of Q. Further conditions are required to determine whether the solution of (j.2.7)$^{\circ}$ is maximizing or minimizing. The continuity conditions (j.2.10) are analogous to the Weierstrass-Erdmann corner conditions.

J.2.3 Necessary conditions for a minimum. Consider defining another scalar function like the Hamiltonian of the previous subarticle (§J.2.2). Denote

$$\text{(j.2.12)} \qquad H' = H + \xi^T\underline{h} + \xi^{*T}\underline{h}$$

That is, H' is the previous Hamiltonian without the control constraints taken into consideration. The functional I of (j.2.1) may be similarly rewritten

$$I' = \iint_Y - H' + \underline{\eta}^T \frac{\partial^2 \underline{x}}{\partial y_1 \partial y_2} dy_1 dy_2$$

$$\text{(j.2.13)} \qquad + \oint_{\partial Y^a} g_a(\underline{x},\sigma)d\sigma + \oint_{\partial Y^b} g_b(\underline{x},\sigma)d\sigma$$

Taking the first variation in I'

$$(j.2.14) \qquad \delta I' = \iint_Y - (\underline{\delta u})^T \frac{\partial H'}{\partial \underline{u}} dy_1 dy_2 \triangleq \iint_Y - \delta H' dy_1 dy_2$$

where $\underline{\eta}$ satisfies $(j.2.7)^o$ with conditions $(j.2.8)^o$ and $(j.2.9)^o$.

For a minimizing control, $\delta I' \geq 0$ (relative minimum) for all admissible $\underline{\delta u}$. This implies $-\delta H' \geq 0$ over Y. Hence for all points on $h_k=0$, $k = m_1+1, \ldots, m$

$$(j.2.15) \qquad \delta H' = (\underline{\delta u})^T \frac{\partial H'}{\partial \underline{u}} \leq 0 \qquad \delta h_k = (\underline{\delta u})^T \frac{\partial h_k}{\partial \underline{u}} \geq 0$$

$$k = m_1 + 1, \ldots, m$$

This implies a relative maximum in H' over all admissible $\underline{u}$ when on the constraint boundary. (The stationary condition on the Hamiltonian of §J.1.1 and §J.1.2 applies away from the constraint boundary.)

The idea for obtaining this result is based on the lumped parameter treatment of Bryson and Ho (1969), Denn (1969) and others. An approach to deriving the statement of the full maximum principle (that is global maximization of the Hamiltonian over the space of admissible controls) could be carried out via the Weierstrass E condition as done by Berkovitz (1961) (lumped parameter case), Lurie (1963) (special type II) among others. The necessary condition of Weierstrass is considered as an analogue of Pontryagin's maximum principle.

<u>J.2.4 A specialization.</u> Consider a system defined over a rectangular domain defined by the limits $[y_j^L, y_j^R]$, $j = 1,2$, then the results of subarticle §J.2.3 assume a particularly simple form. Formally, given a system modelled according to a type III form

$$(j.1.1) \qquad \frac{\partial^2 \underline{x}}{\partial y_1 \partial y_2} = \underline{f}[\underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}, y_1, y_2)$$

with end-state conditions

(j.1.1a)" $x_i(y_1, y_2), \ldots, \partial_{\ell-1}x_i(y_1, y_2)$ specified at y_j^L, y_j^R; $j = 1,2$

($\ell=\ell_1$ on y_1^L, y_1^R boundaries, $\ell=\ell_2$ on y_2^L, y_2^R boundaries. Split boundary conditions are implied), and criterion

(j.1.3)"
$$Q = \iint_Y G[\underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}, y_1, y_2]\,dy_1dy_2 + \int_{y_1} g_1(\underline{x}, y_1)dy_1 \Bigg|_{y_2^L}^{y_2^R} + \int_{y_2} g_2(\underline{x}, y_2)dy_2 \Bigg|_{y_1^L}^{y_1^R}$$

the minimizing control vector $\underline{u}(y_1, y_2)$ is chosen from some admissible set U such that the first variation in the Hamiltonian

$$H' = \underline{\eta}(y_1, y_2)^T \underline{f}[\underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}, y_1, y_2] - G[\underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}, y_1, y_2]$$

is less than or equal to zero (that is relative maximality of H' with respect to $\underline{u}$ when on a control constraint boundary). For admissible controls away from the constraint boundary the first derivative of H' with respect to u gives the stationary value of Q.

The vector function $\underline{\eta}(y_1, y_2)$ is found from the 'adjoint equations'

(j.2.7a)"
$$\frac{\partial^2 \underline{\eta}}{\partial y_1 \partial y_2} = \frac{\partial H}{\partial \underline{x}} + (-1)^\ell \partial_\ell \left[\frac{\partial H}{\partial [\partial_\ell \underline{x}]}\right]$$

with boundary conditions

(i) at y_1^L, y_1^R

(j.2.8)"
$$x_i \text{ given or } \frac{\partial \eta_i}{\partial y_2} + (-1)^{\ell_1 - 1}\partial_{\ell_1 - 1}\left[\frac{\partial H}{\partial [\partial_{\ell_1} x_i]}\right] - \frac{\partial g_2}{\partial x_i} = 0$$

$$i = 1, \ldots, n$$

(ii) at y_2^L, y_2^R

$$(j.2.9)" \qquad x_i \text{ given or } \frac{\partial \eta_i}{\partial y_1} + (-1)^{\ell_2-1}\partial_{\ell_2-1}\left[\frac{\partial H}{\partial[\partial_{\ell_2}x_i]}\right] - \frac{\partial g_1}{\partial x_i} = 0$$

$$i = 1, \ldots , n$$

Also

$$(j.2.7d)" \qquad \text{at } (y_1^L, y_2^L), (y_1^R, y_2^R) \quad x_i \text{ given or } \eta_i = 0, \; i = 1 , \ldots , n.$$

Equations (j.1.1) and (j.2.7a)" may be solved simultaneously for the $2n+r$ unknowns $x_1, \ldots , x_n, \eta_1, \ldots , \eta_n, u_1, \ldots , u_r$. The boundary conditions are of the split type; for $\underline{x}$ they are (j.1.1a)"; for $\underline{\eta}$ they are the differential equations (j.2.8)" and (j.2.9)" which themselves have boundary conditions (j.2.7d)" (- equations (j.2.8)" and (j.2.9)" with conditions (j.2.7d)" uniquely define $\underline{\eta}$ along the boundaries for any given control). The coupling relations between the state equations (j.1.1) and the adjoint equations (j.2.7a)" are provided by the nonpositivity condition on the first variation (or the first derivative condition) of the Hamiltonian given by (j.2.12).

The above simplification of the results of article §J.2.3 only apply when the differentials dy_1 and dy_2 appearing in $(j.2.8)^o$ and $(j.2.9)^o$ are the same. This can only be achieved for a region with inner and outer boundaries (aligned with the coordinate axes directions) when both inner and outer boundaries are squares. In transferring to a region with only an outer boundary as in this subarticle, the results are directly applicable where this boundary is a square; when the boundary is rectangular, an inner imaginary line boundary parallel to the long side should be included so that increments in the y_1 and y_2 directions are equal - boundary conditions on this imaginary line boundary are continuity conditions on the state across the boundary (see §H). The postulate of the imaginary inner boundary may of course be removed if the necessary conditions are derived directly for the rectangular simply connected region, in which case the results are the same as given in this subarticle.

J.3 DISCUSSION

As in previous sections, it is not required to derive the proof each time an optimum design is contemplated; the essential results are contained in the summary statements (§J.2.2, 4). The complete solution involves solving a boundary value problem, composed of the state and adjoint equations which are analogous to the Hamilton canonic equations of the lumped parameter case. The two sets of equations are related by an extremum condition on the Hamiltonian yielding a control vector as a function of the state and adjoint vectors. Comments on the characteristics of these equations, given in §F.4, are valid here. Their solution, essentially because they contain second order derivatives on the left hand sides, may however be more awkward. The maximum principle represents a systematic solution technique for design problems with its ability to give a general prescription of the solution, to a particular design problem or a whole class of design problems, being its main attribute. The conditions are necessary and have to be satisfied by each extremal solution; the extremal solutions are not given directly as noted previously.

The derivation shows a connection between the calculus of variations and the maximum principle of Pontryagin (a previous section §H shows the relationship of the maximum principle to the continuous form of dynamic programming). Using the calculus of variations arguments, the necessary conditions for the control to produce a stationary value in the criterion were found to be that the derivative of the Hamiltonian with respect to the control be zero for all $\underline{y}$. This was extended by considering further the first variation in the criterion, to lead to the necessary requirement of the nonpositivity of the first variation in the Hamiltonian for the control to be optimal. Both necessary conditions are local with the latter implying a local maximization of H over the admissible region U. This is compared with a statement of the complete maximum principle which implies a global maximization of H over the admissible region U.

The derivation highlights the essential difference between the maximum and minimum principle forms. Assuming a Hamiltonian of opposite sign to H' had been chosen, then the first variation $\delta H'$ given by equation (j.2.15) would be nonnegative (rather than nonpositive) implying a relative minimum (rather than a maximum). Continuing this comment further, if the substitution

$$\frac{\partial^2 x_o}{\partial y_1 \partial y_2} = f_o = G[\underline{x}, \ldots, \partial_\ell \underline{x}, \ldots, \underline{u}, y_1, y_2]$$

with $x_o(0, y_2) = 0$, $x_o(y_1, 0) = 0$ ($\Rightarrow x_o(0, 0) = 0$) is made, equation (h.2.7a)" becomes

$$\frac{\partial^2 \underset{\sim}{\eta}}{\partial y_1 \partial y_2} = \frac{\partial H}{\partial \underset{\sim}{x}} + (-1)^\ell \partial_\ell \left[\frac{\partial H}{\partial [\partial_\ell \underset{\sim}{x}]}\right]$$

(j.3.1)

$$= \sum_{j=0}^{n} \left\{\eta_j \frac{\partial f_j}{\partial \underset{\sim}{x}} + (-1)^\ell \eta_j \partial_\ell \left[\frac{\partial f_j}{\partial [\partial_\ell \underset{\sim}{x}]}\right]\right\}$$

where the tilda notation implies n+1 component vectors, such as $\underset{\sim}{\eta} = (\eta_o, \eta_1, \ldots, \eta_n)^T$. Equation (j.3.1) can be seen to be homogeneous and linear in η_j; that is if $\alpha(y_1, y_2)$ is a solution of (j.3.1), then $-\alpha(y_1, y_2)$ is also a solution. And if it is noted that the Lagrange multipliers $\underline{\eta}$ are of opposite sign in H' (after including η_o multiplying G) and the negative of H' mentioned above, then the maximum and minimum principle forms are equivalent. Generally, in more thorough treatments of the maximum principle (for example Pontryagin et al 1962, Leitmann 1966, Boltyanskii 1971), η_o is taken to be any nonpositive value (usually -1) and the remaining η_j, j = 1, ... , n scaled to suit; in the minimum principle η_o is taken to be any nonnegative value (usually +1) and the remaining η_j, j = 1, ... , n scaled to suit. It will be noted from (j.3.1) that

$$\frac{\partial^2 \eta_o}{\partial y_1 \partial y_2} = 0$$

and hence η_o is constant for all (y_1, y_2).

An illustration of the use of the necessary conditions follows. It demonstrates the theoretical problems which arise and the mathematical procedure for handling them.

§K A DESIGN ILLUSTRATION: SYSTEM MODEL TYPE III

K.1 GENERAL

This section details and solves the plate illustration using a type III model and the associated optimality conditions derived in the previous section (§J). Previous sections consider the design using a type I model (§G) and type II model (§I). It is shown that the optimality conditions of §J when applied to the same problem reduce to the same equations as those given in §G and §I.

Broadly, the illustration problem entails the minimum weight design of a freely vibrating plate with a constraint on a fundamental frequency. In terms of the original plate symbols, an explicit statement may be found in §G and §I. As in §G and §I, the intention of the illustration is to outline the theoretical problems which arise in design using the maximum principle and the mathematical procedures for handling them.

Several reduction procedures (that is several choices of the set of states) are outlined and discussed. In one case the state equations are shown to reduce to the fundamental set of equations considered by A.I. Egorov (1964). However to achieve such a result it is found necessary to introduce auxiliary dependent (control) variables analogous to those entailed in reductions to a type II format. In all cases, the design computations are found to be surprisingly uncluttered when compared with the designs of the previous sections. The approach to this section is to outline the computations fully for one possible reduction and then to show the modifications available following this solution.

The control in this problem, as occurred wholly or partly in the previous illustrations, occurs linearly in the Hamiltonian. This singular condition is by-passed, pending a more formal treatment of singularities in §L, by a suitable substitution that implicitly satisfies the conditions for optimality in the presence of singularities (as given in §L).

A numerical solution of the problem and the rudiments of a sensitivity analysis of the solution are given in §G and §I respectively.

K.2 TRANSFORMATION TO A TYPE III FORMAT

Essentially two distinct approaches as outlined in part 1 are possible. Firstly, the state may be chosen as the displacement, twisting moment pair and the control as the plate rigidity. Such a choice is intuitively appealing but reduces to a type III form only for the case of constant geometry. Secondly (after appendix one), and the reduction adopted here, the derivatives in the high order system equation are expanded first and control and states defined to satisfy the mathematical requirements of the type III format. This leads to a higher number of states and state equations.

Differentiations involving mixed-derivative terms in equation (g.2.3), ((i.1.1)) are carried out to give an expanded form of this equation with isolated mixed-derivative terms;

$$\frac{\partial^2}{\partial y_1^2}\left(D\frac{\partial^2 W}{\partial y_1^2}\right) + \frac{\partial^2}{\partial y_2^2}\left(D\,\frac{\partial^2 W}{\partial y_2^2}\right) + \nu\left(\frac{\partial^2 D}{\partial y_1^2}\,\frac{\partial^2 W}{\partial y_2^2} + \frac{\partial^2 D}{\partial y_2^2}\,\frac{\partial^2 W}{\partial y_1^2}\right)$$

$$(k.2.1) \qquad + 2\left(\frac{\partial D}{\partial y_1}\,\frac{\partial^3 W}{\partial y_1 \partial y_2^2} + \frac{\partial D}{\partial y_2}\,\frac{\partial^3 W}{\partial y_1^2 \partial y_2} + D\,\frac{\partial^4 W}{\partial y_1^2 \partial y_2^2}\right)$$

$$+ 2(1-\nu)\left(\frac{\partial^2 D}{\partial y_1 \partial y_2}\,\frac{\partial^2 W}{\partial y_1 \partial y_2}\right) - e^2 D^{\frac{1}{3}} W = 0$$

Introduce state and control notation according to

$$(k.2.2) \qquad \begin{bmatrix} x_1 \triangleq W \\ \\ x_2 \triangleq \dfrac{\partial^2 W}{\partial y_1 \partial y_2} \\ \\ x_3 \triangleq D \end{bmatrix}$$

$$u \triangleq \frac{\partial^2 D}{\partial y_1 \partial y_2}$$

Taking the mixed $y_1 y_2$ derivatives of the states leads to the system equations

$$(k.2.3)\qquad \begin{bmatrix} \dfrac{\partial^2 x_1}{\partial y_1 \partial y_2} = x_2 \\[2ex] \dfrac{\partial^2 x_2}{\partial y_1 \partial y_2} = \dfrac{-\theta}{2x_3} \\[2ex] \dfrac{\partial^2 x_3}{\partial y_1 \partial y_2} = u \end{bmatrix}$$

where

$$\theta = \left\{\frac{\partial^2}{\partial y_1^2}\left(x_3 \frac{\partial^2 x_1}{\partial y_1^2}\right) + \frac{\partial^2}{\partial y_2^2}\left(x_3 \frac{\partial^2 x_1}{\partial y_2^2}\right)\right.$$

$$+\nu\left(\frac{\partial^2 x_3}{\partial y_1^2}\frac{\partial^2 x_1}{\partial y_2^2} + \frac{\partial^2 x_3}{\partial y_2^2}\frac{\partial^2 x_1}{\partial y_1^2}\right) + 2\left(\frac{\partial x_3}{\partial y_1}\frac{\partial x_2}{\partial y_2} + \frac{\partial x_3}{\partial y_2}\frac{\partial x_2}{\partial y_1}\right)$$

$$\left. +2(1-\nu)ux_2 - e^2 x_3^{\frac{1}{3}} x_1\right\}$$

Equations (k.2.3) are now in the standard form (c.3.3)

$$\frac{\partial^2 \underline{x}}{\partial y_1 \partial y_2} = \underline{f}[\underline{x}, \ \ldots \ , \partial_\ell \underline{x}, \ \ldots \ , \underline{u}, y_1, y_2]$$

where $\underline{x} = (x_1, x_2, x_3)^T$, $\underline{u} = u$.

State boundary conditions associated with (k.2.3) become

$$(k.2.3a)\qquad \begin{aligned} x_1\Big|_{y_1 = 0,a} &= 0 \qquad & x_3 \frac{\partial^2 x_1}{\partial y_1^2} + \nu x_3 \frac{\partial^2 x_1}{\partial y_2^2}\Bigg|_{y_1 = 0,a} &= 0 \\ x_1\Big|_{y_2 = 0,b} &= 0 \qquad & x_3 \frac{\partial^2 x_1}{\partial y_2^2} + \nu x_3 \frac{\partial^2 x_1}{\partial y_1^2}\Bigg|_{y_2 = 0,b} &= 0 \end{aligned}$$

K.3 DESIGN COMPUTATIONS.

The problem is to give optimality to

$$(k.3.1)\qquad Q = \int_{y_1}\int_{y_2} x_3^{\frac{1}{3}} dy_2 dy_1$$

(in terms of the introduced variables above) for the system behaving according to (k.2.3) and (k.2.3a).

The Hamiltonian is then of the form

$$\text{(k.3.2)} \qquad H = -x_3^{\frac{1}{3}} + \eta_1 x_2 - \frac{\eta_2 \theta}{2x_3} + \eta_3 u$$

It is remarked that H is linear in u and without prespecified constraints on the control, the situation corresponds to a singular formulation. (See §L for comments.) The maximum principle requires that H be maximized over u as a necessary condition for optimality. Following arguments similar to the illustration in §G a substitution for the adjoint variables will be made which implicitly satisfies the requirements for optimality in the presence of singularities using the maximum principle. For the present, the coefficient of u in the Hamiltonian is

$$\text{(k.3.3)} \qquad \sigma(y_1, y_2) = -\frac{(1-\nu)x_2\eta_2}{x_3} + \eta_3$$

The adjoint variables η_i, i = 1,2,3 are defined by

$$\text{(k.3.4)} \qquad \left[\begin{aligned}
\frac{\partial^2 \eta_1}{\partial y_1 \partial y_2} &= \frac{\partial^2}{\partial y_1^2}\left(x_3 \frac{\partial^2 \phi}{\partial y_1^2}\right) + \frac{\partial^2}{\partial y_2^2}\left(x_3 \frac{\partial^2 \phi}{\partial y_2^2}\right) \\
&\quad + \nu \frac{\partial^2}{\partial y_2^2}\left(\phi \frac{\partial^2 x_3}{\partial y_1^2}\right) + \nu \frac{\partial^2}{\partial y_1^2}\left(\phi \frac{\partial^2 x_3}{\partial y_2^2}\right) - \phi e^2 x_3^{\frac{1}{3}} \\
\frac{\partial^2 \eta_2}{\partial y_1 \partial y_2} &= \eta_1 - 2\frac{\partial}{\partial y_2}\left(\phi \frac{\partial x_3}{\partial y_1}\right) - 2\frac{\partial}{\partial y_1}\left(\phi \frac{\partial x_3}{\partial y_2}\right) + 2(1-\nu)\phi u \\
\frac{\partial^2 \eta_3}{\partial y_1 \partial y_2} &= -\frac{1}{3} x_3^{-\frac{2}{3}} - \frac{\phi\theta}{x_3} + \frac{\partial^2 \phi}{\partial y_1^2}\frac{\partial^2 x_1}{\partial y_1^2} + \frac{\partial^2 \phi}{\partial y_2^2}\frac{\partial^2 x_1}{\partial y_2^2} \\
&\quad + \nu \frac{\partial^2}{\partial y_1^2}\left(\phi \frac{\partial^2 x_1}{\partial y_2^2}\right) + \nu \frac{\partial^2}{\partial y_2^2}\left(\phi \frac{\partial^2 x_1}{\partial y_1^2}\right)
\end{aligned}\right.$$

$$\left. - 2 \frac{\partial}{\partial y_1}\left(\phi \frac{\partial x_2}{\partial y_2}\right) - 2 \frac{\partial}{\partial y_2}\left(\phi \frac{\partial x_3}{\partial y_1}\right) - \frac{1}{3} e^2 \phi x_3^{-\frac{2}{3}} x_1 \right]$$

where $\phi = \dfrac{-\eta_2}{2x_3}$

with natural boundary conditions,
along $y_1 = 0,a$

$$\left[\frac{\partial \eta_2}{\partial y_2} + 2 \frac{\partial x_3}{\partial y_2} \phi\right] \delta x_2 = 0$$

(k.3.4a)
$$\left[\frac{\partial \eta_3}{\partial y_2} - \frac{\partial}{\partial y_1}\left(\phi \frac{\partial^2 x_1}{\partial y_1^2}\right) + 2\phi \frac{\partial^3 x_1}{\partial y_1^3} \right.$$
$$\left. - \nu \frac{\partial}{\partial y_1}\left(\phi \frac{\partial^2 x_1}{\partial y_2^2}\right) + 2\phi \frac{\partial x_2}{\partial y_2}\right] \delta x_3 = 0$$

along $y_2 = 0,b$

$$\left[\frac{\partial \eta_2}{\partial y_1} + 2 \frac{\partial x_3}{\partial y_1} \phi\right] \delta x_2 = 0$$

(k.3.4b)
$$\left[\frac{\partial \eta_3}{\partial y_1} - \frac{\partial}{\partial y_2}\left(\phi \frac{\partial^2 x_1}{\partial y_2^2}\right) + 2\phi \frac{\partial^3 x_1}{\partial y_2^3} \right.$$
$$\left. - \nu \frac{\partial}{\partial y_2}\left(\phi \frac{\partial^2 x_1}{\partial y_1^2}\right) + 2\phi \frac{\partial x_2}{\partial y_1}\right] \delta x_3 = 0$$

with $\eta_2(0,0) = \eta_2(a,b) = \eta_3(0,0) = \eta_3(a,b) = 0$

Conditions (k.3.4a,b) are differential equations in $\underline{\eta}$ with given initial and final conditions, from which $\underline{\eta}$ may be found along the boundaries, for given control.

An optimal solution is thus required to satisfy the boundary value problem expressed by the simultaneous equations (k.2.3) and (k.3.4) with boundary conditions (k.2.3a) and (k.3.4a,b,c). There is a total of six equations in seven unknowns x_1, x_2, x_3, η_1, η_2, η_3 and u. Additionally the coefficient σ (expression (k.3.3)) is required to be maintained at zero over the optimal region; this gives the extra equation needed to completely solve the problem. An alternative approach, analogous to the previous illustrations in §G and §I, to the solution however, will be used here.

By inspection of equations (k.2.3) and (k.3.4), a valid substitution which satisfies the boundary conditions of (k.3.4a,b,c) as well as equations (k.3.4) is

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = -2\gamma^2 \begin{bmatrix} x_3x_2 - \nu x_1 u \\ x_1x_3 \\ (1-\nu)x_1x_2 \end{bmatrix} \tag{k.3.5}$$

The substitution also maintains (over the total plate region) $\sigma = 0$.

Using (k.3.5), equation (k.3.4)[1] reduces to the original system equation (k.2.1) when expressed in the original plate variables, equation (k.3.4)[2] reduces to an identity, and equation (k.3.4)[3] reduces to

$$\begin{aligned} &\frac{\partial^2 w}{\partial y_1^2}\left(\frac{\partial^2 w}{\partial y_1^2} + \nu\,\frac{\partial^2 w}{\partial y_2^2}\right) + 2(1-\nu)\left(\frac{\partial^2 w}{\partial y_1 \partial y_2}\right)^2 \\ &+ \frac{\partial^2 w}{\partial y_2^2}\left(\frac{\partial^2 w}{\partial y_2^2} + \nu\,\frac{\partial^2 w}{\partial y_1^2}\right) - \frac{1}{3\gamma^2}\left[1 + \gamma^2 e^2 w^2\right] D^{-\frac{2}{3}} = 0 \end{aligned} \tag{k.3.6}$$

The relevant boundary conditions are (i.1.1a) (or g.2.8). That is, for an optimal solution equation (k.3.6) has to be solved simultaneously with (k.2.1). However these are in fact the same equations determined in §G and §I for types I and II system formats respectively. All approaches consequently reduce to the same equations to be solved. By graphical means it can be shown that the solution for u obtained not only leads to a stationary value for H but is also maximizing and hence optimal according to the maximum principle.

K.4 ALTERNATIVE CHOICES OF STATES

The order of equations (k.2.3) may be reduced if additional state variables are defined. A reduced order may be helpful in certain computational situations. Consider the set

$$\text{(k.4.1)}\qquad \begin{bmatrix} \xi_1 \triangleq W \\ \\ \xi_2 \triangleq \dfrac{\partial W}{\partial y_1} \\ \\ \xi_3 \triangleq \dfrac{\partial W}{\partial y_2} \\ \\ \xi_4 \triangleq \dfrac{\partial^2 W}{\partial y_1 \partial y_2} \\ \\ \xi_5 \triangleq D \end{bmatrix}$$

with control, $\mu \triangleq \dfrac{\partial^2 D}{\partial y_1 \partial y_2}$

From these, the state equations are

$$\text{(k.4.2)}\qquad \begin{bmatrix} \dfrac{\partial^2 \xi_1}{\partial y_1 \partial y_2} = \xi_4 \\ \\ \dfrac{\partial^2 \xi_2}{\partial y_1 \partial y_2} = \dfrac{\partial \xi_4}{\partial y_1} \\ \\ \dfrac{\partial^2 \xi_3}{\partial y_1 \partial y_2} = \dfrac{\partial \xi_4}{\partial y_2} \\ \\ \dfrac{\partial^2 \xi_4}{\partial y_1 \partial y_2} = \dfrac{-\theta}{2\xi_5} \\ \\ \dfrac{\partial^2 \xi_5}{\partial y_1 \partial y_2} = \mu \end{bmatrix}$$

where $$\theta = \frac{\partial^2}{\partial y_1^2}\left(\xi_5 \frac{\partial \xi_2}{\partial y_1}\right) + \frac{\partial^2}{\partial y_2^2}\left(\xi_5 \frac{\partial \xi_3}{\partial y_2}\right) + \nu\left(\frac{\partial^2 \xi_5}{\partial y_1^2}\frac{\partial \xi_3}{\partial y_2} + \frac{\partial^2 \xi_5}{\partial y_2^2}\frac{\partial \xi_2}{\partial y_1}\right)$$

$$+ 2\left(\frac{\partial \xi_5}{\partial y_1}\frac{\partial \xi_4}{\partial y_2} + \frac{\partial \xi_5}{\partial y_2}\frac{\partial \xi_4}{\partial y_1}\right) + 2(1-\nu)\mu\xi_4 - e^2 \xi_5^{\frac{1}{3}} \xi_1$$

which are again of the general form (c.3.3)

$$\frac{\partial^2 \underline{\xi}}{\partial y_1 \partial y_2} = \underline{f}'[\underline{\xi}, \ldots, \partial_\ell \underline{\xi}, \ldots, \underline{\mu}, y_1, y_2]$$

where $\underline{\xi} = (\xi_1, \ldots, \xi_5)^T$, $\underline{\mu} = \mu$.

Also by using the device employed in connection with the type II systems, namely introducing auxiliary dependent (control) variables, the general system form of A.I. Egorov (1964) and Butkovskii (1969) may be obtained. In particular define the following states

(k.4.3a) $$\begin{bmatrix} \rho_1 \triangleq W \\ \rho_2 \triangleq \dfrac{\partial W}{\partial y_1} \\ \rho_3 \triangleq \dfrac{\partial W}{\partial y_2} \\ \rho_4 \triangleq \dfrac{\partial^2 W}{\partial y_1 \partial y_2} \\ \rho_5 \triangleq \dfrac{\partial^2 W}{\partial y_1^2} \\ \rho_6 \triangleq \dfrac{\partial^2 W}{\partial y_2^2} \\ \rho_7 \triangleq D \end{bmatrix}$$

and control $\kappa \triangleq \dfrac{\partial^2 D}{\partial y_1 \partial y_2}$, with auxiliary dependent (control) variables the remaining second order derivatives of D and fourth order derivatives of W except

$$\frac{\partial^4 W}{\partial y_1^2 \partial y_2^2} \left(= \frac{\partial^2 \rho_4}{\partial y_1 \partial y_2}\right) ;$$

(k.4.3b)
$$\kappa_1 \triangleq \frac{\partial^4 W}{\partial y_1^4} , \quad \kappa_2 \triangleq \frac{\partial^4 W}{\partial y_1^3 \partial y_2} , \quad \kappa_3 \triangleq \frac{\partial^4 W}{\partial y_1 \partial y_2^3} , \quad \kappa_4 \triangleq \frac{\partial^4 W}{\partial y_2^4}$$

$$\kappa_5 \triangleq \frac{\partial^2 D}{\partial y_1^2} , \quad \kappa_6 \triangleq \frac{\partial^2 D}{\partial y_2^2}$$

Differentiating the state variables $\underline{\rho}$ leads to

(k.4.4)
$$\begin{bmatrix} \dfrac{\partial^2 \rho_1}{\partial y_1 \partial y_2} = \rho_4 \\[2ex] \dfrac{\partial^2 \rho_2}{\partial y_1 \partial y_2} = \dfrac{\partial \rho_4}{\partial y_1} \\[2ex] \dfrac{\partial^2 \rho_3}{\partial y_1 \partial y_2} = \dfrac{\partial \rho_4}{\partial y_2} \\[2ex] \dfrac{\partial^2 \rho_4}{\partial y_1 \partial y_2} = -\dfrac{\theta}{2\rho_7} \\[2ex] \dfrac{\partial^2 \rho_5}{\partial y_1 \partial y_2} = \kappa_2 \\[2ex] \dfrac{\partial^2 \rho_6}{\partial y_1 \partial y_2} = \kappa_3 \\[2ex] \dfrac{\partial^2 \rho_7}{\partial y_1 \partial y_2} = \kappa \end{bmatrix}$$

where $$\theta = \kappa_5\rho_5 + 2\frac{\partial\rho_7}{\partial y_1}\frac{\partial\rho_5}{\partial y_1} + \rho_7\kappa_1 + \kappa_6\rho_6 + 2\frac{\partial\rho_7}{\partial y_2}\frac{\partial\rho_6}{\partial y_2} + \rho_7\kappa_4$$

$$+ \nu(\kappa_5\rho_6 + \kappa_6\rho_5) + 2\left(\frac{\partial\rho_7}{\partial y_1}\frac{\partial\rho_6}{\partial y_1} + \frac{\partial\rho_7}{\partial y_2}\frac{\partial\rho_5}{\partial y_2}\right)$$

$$+ 2(1-\nu)\kappa\rho_4 - e^2\rho_7^{\frac{1}{3}}\rho_1$$

which are now in the form

$$\text{(k.4.5)} \qquad \frac{\partial^2\underline{\rho}}{\partial y_1\partial y_2} = \underline{f}''\left[\underline{\rho}, \frac{\partial\underline{\rho}}{\partial y_1}, \frac{\partial\underline{\rho}}{\partial y_2}, \underline{\kappa}, y_1, y_2\right]$$

where $\underline{\rho} = (\rho_1, \ldots, \rho_7)^T$, $\underline{\kappa} = (\kappa, \kappa_1, \ldots, \kappa_6)^T$. This is the general form treated by A.I. Egorov (1964) and Butkovskii (1969) where state derivatives up to the first order are allowed on the right hand side.

It is remarked that the choice of state variables (k.4.1) leads to the system equations (k.4.2) which contain the lowest order state derivatives on the right hand side (of system type III form for this structure) without resorting to the device of introducing the auxiliary control variables. In both reductions (k.4.1) and (k.4.3) the number of state equations ((k.4.2) and (k.4.4) respectively) is naturally increased. Following the same computations path as in the previous article (§K.3), it can be shown for the two latest proposed reductions, that the same equations ((k.2.1) and (k.3.6)) have to be solved for optimality.

K.5 COMMENT

For the particular illustration chosen, employing a system model type III, very economical computations resulted. It would be anticipated that type III would be more suitable than type I, essentially because the structure and problem exhibit symmetric behaviour, whereas the type I model is fundamentally unidirectional in nature. However it is unclear why type III appears more economical than type II; in the former case both the number of variables and equations are less, while auxiliary controls were unnecessary but could be included. (It may be argued though that type II produces lower order equations which may be advantageous in many computational situations.)

Comments regarding the qualities of the maximum principle as a design tool, its range of application and its computational limitations given in §G and §I, are valid here.

By way of interest it is noted that in the state equations (k.2.3), (k.4.2) and (k.4.4) particular state variables have occurred on the right hand side of the state equation referring to that state variable. (For example, x_2 occurs on both the right and left hand sides of $(k.2.3)^2$.) For the chosen illustration this does not occur with types I and II.

§L SINGULAR SOLUTIONS IN DESIGN

L.1 INTRODUCTION

L.1.1 Outline. A natural extension of the application of the foregoing conditions for optimality of sections §F, §H and §J is to examine the situation under which a solution may be singular, that is where the maximum principle and the necessary conditions of the calculus of variations are trivially satisfied and fail to give the optimum solution directly. The occurrence of singular solutions is shown in typical problems of structural design and hence their study is of more than academic interest. (The situation has already arisen in sections §G, §I and §K but formal treatment was avoided by employing suitable substitutions.) In this section, a certain type of singular solution of control problems is investigated, the type being characteristic of these above mentioned structural design problems. The engineering implications of singular controls are considered.

L.1.2 Background. The singular problem in optimum structural design does not appear to have been studied previously although the environment for creating singular formulations has been present with the use of the calculus of variations as a design tool. It appears that the conditions leading to the singular problem would have existed in past applications of the calculus of variations as general transformations are available (see for example Berkovitz 1961, Leitmann 1966) between problems with a control format and the calculus of variations. In the structures literature, de Silva (1972) and Armand (1971, 1972) give design problem formulations which on application of Pontryagin's principle only result in extremal solutions and are not necessarily minimizing. Nevertheless, as shown in this section, their results may be strengthened and shown to be also optimal.

In the control literature, a growing body of material is available for handling the singular problem, material from which the singular structural design problem may now draw as a result of the modelling procedures outlined in part 1 and the design techniques of this part. Contributions to the singular problem phenomenon in control have been by Kelley (1964), Hermes (1964), Johnson (1965), Johnson and Gibson (1963), Miele (1962a), Hermes and Haynes (1963) and Kopp and Moyer (1965) among others. In addition many authors have discussed the subject. (See for example Rozonoer 1959, Pontryagin et al 1962, Leitmann 1966.) Individual contributions are noted in the following discussions when reference is made to their results.

L.2 THE SINGULAR PROBLEM

L.2.1 Characteristics. From section §F it will be recalled that for the system described by

$$(\ell.2.1) \qquad \frac{\partial x_i}{\partial y_4} = \frac{\partial H}{\partial \lambda_i} \qquad i = 1, \ldots , n$$

with adjoint equations

$$(\ell.2.2) \qquad \frac{\partial \lambda_i}{\partial y_4} = - \frac{\partial H}{\partial x_i} - (-1)^L \, \partial_{\underline{\ell}} \left[\frac{\partial H}{\partial [\partial_{\underline{\ell}} \, x_i]}\right] \qquad i = 1, \ldots , n$$

and certain end-state, natural boundary and transversality conditions, where $\underline{\ell} = (\ell_1, \ell_2, \ell_3)$ and $\partial_{\underline{\ell}}[\cdot]$ is as defined in the 'notation', the optimal control is chosen so as to maximise the Hamiltonian. Formally,

$$(\ell.2.3) \qquad H(\underline{y}, \underline{x}, \ldots , \partial_{\underline{\ell}}\underline{x}, \ldots , \underline{\hat{u}}, \underline{\Pi}) \geq H(\underline{y}, \underline{x}, \ldots , \partial_{\underline{\ell}}\underline{x}, \ldots , \underline{u}, \underline{\Pi})$$
$$\underline{u} \in U , \underline{y} \in Y$$

A similar result was obtained for type II systems in section §H, and type III systems in §J.

However in certain instances this inequality reduces to an equality, in which case the maximum principle fails to give the optimal control. If H is linear in one or more of the control components u_i $(i = 1, \ldots , r)$. Then $\partial H/\partial u_i$ is independent of u_i and $\partial^2 H/\partial u_i \partial u_j = 0$; $i, j = 1, \ldots , r$. (Setting $\partial H/\partial u_i = 0$ defines an extremal solution but where the matrix $\partial^2 H/\partial u_i \partial u_j = 0$ for the same solution, it is unknown whether the solution is optimal or not.) In this case H and $\partial H/\partial u_i$ may vanish over a finite region of the independent variables $\underline{y}$, depending on the construction of the coefficient of u_i in the expression for H, and no further may be said about the optimal u_i. The maximum principle is unable to find the optimum control in this case. The situation in which the linear control coefficient (equivalently $^{\partial H}/_{\partial u_i}$) vanishes is referred to as singular (see for example Johnson and Gibson 1963). The structures problems of Armand (1971, 1972) and de Silva (1972) are of this form. Singular solutions may or may not form part or whole of the optimal solution; optimality has to be shown. The behaviour of singular solutions will be studied in detail with reference to one of Armand's problems. Attention will be restricted to lumped parameter problems of Armand's form in one control variable where the Hamiltonian is linear in the control but nonlinear in the state.

The solution of the general singular problem can be seen to be governed by the values taken by the coefficient of the linear control term appearing in the Hamiltonian. For a system constrained according to $u_{min}(y) \leq u(y) \leq u_{max}(y)$, where at any y, u_{min} and u_{max} are given bounds determined by U, the control assumes the maximum bound when the coefficient is positive and the minimum bound when the coefficient is negative. Formally

$$\hat{u}(y) = \begin{cases} u_{min}(y) & \text{for } \sigma(y) < 0 \\ u_{max}(y) & \text{for } \sigma(y) > 0 \end{cases}$$

where $\sigma(y)$ is the coefficient of the linear control term, often called a 'switching function' (see for example Leitmann 1966). This in principle, creates a well-defined piecewise continuous ('bang-bang') control $\hat{u}(y)$, the assumption being that the coefficient σ becomes zero at only isolated values of $y \in [y^L, y^R]$. However, the coefficient may vanish over a finite subinterval of the interval $[y^L, y^R]$. The corresponding control is termed singular and is not well-defined. A singular control may comprise a subarc of the optimal control. The optimal control may then not be unique - in fact it is only extremal. (This is in addition to the possibility that the solution by the maximum principle may have not been unique to start with - see section §F.) The nonsingular portions (corresponding to the direct solution of (ℓ.2.1), (ℓ.2.2) and (ℓ.2.3) with $\hat{u} = u_{max}$ or u_{min}) of the extremal control are defined by the boundary conditions and certain continuity properties with the singular portions. (The singular portions are still required to satisfy the system and adjoint equations.) The possible choices of extremal controls are consequently reduced, the optimal giving the least value of the criterion. It is remarked that the presence of a singularity in the solution need not necessarily imply that the optimal solution contains a singular portion. This has to be shown. The appearance of singularities thus involves definite analytical difficulties.

Note that conditions other than those outlined here (the present problems are linear in one or more of the controls but nonlinear in one or more of the states) may lead to singular solutions. The reader is referred to, for example Kelley, Kopp and Moyer (1966), Johnson (1965), for a discussion on this. The conditions here are the only known form to have occurred in structural applications to date.

L.2.2 The solution of singular problems. A singular solution may be found from the property that the coefficient of the control remains zero on the singular arc, or equivalently from the vanishing of the coefficient's derivatives with respect to the independent variable y;

$$(\ell.2.4) \qquad \frac{d}{dy}\left(\frac{\partial H}{\partial u}\right) = \frac{d^2}{dy^2}\left(\frac{\partial H}{\partial u}\right) = \frac{d^3}{dy^3}\left(\frac{\partial H}{\partial u}\right) = \ldots = 0$$

That is, u is determined such that $\sigma = \frac{\partial H}{\partial u} = 0$ over the particular interval of interest. Each derivative is applied successively until an expression containing the control is obtained. Use is made of the system and adjoint equations to express the singular control in terms of the state and adjoint variables

$$u(y) = u(\underline{x}(y), \underline{\Pi}(y), y)$$

To determine whether a singular arc is optimal (recalling that $\frac{\partial^2 H}{\partial u^2} = 0$ and thus no conclusion can be drawn from this test applicable for nonsingular arcs), an additional necessary condition to equation (ℓ.2.4) above, analogous to ensuring that the second derivative of the Hamiltonian is strictly positive, has been derived by Kelley (1964), Kopp and Moyer (1965), Kelley, Kopp and Moyer (1966) and Robbins (1965). By considering a special perturbation in the singular control, they show that

$$(\ell.2.5) \qquad (-1)^{\frac{\kappa+2}{2}} \frac{\partial}{\partial u}\left[\frac{d^\kappa}{dy^\kappa}\left(\frac{\partial H}{\partial u}\right)\right] > 0 \qquad \kappa = 0, 2, 4, \ldots$$

must be satisfied along a singular arc for a minimum Q. The index κ represents the smallest order ($\geq$ 2) derivative of $\sigma = \frac{\partial H}{\partial u}$ with respect to y, which is an explicit function of u. The necessary condition is applied for increasing values of κ until the right side differs from zero. When the right hand side equals zero, the result is inconclusive. Kelley, Kopp and Moyer (1966) show that if $\frac{\partial H}{\partial u}$ is successively differentiated with respect to y, then u cannot first explicitly appear in an odd order derivative. No conditions expressing the sufficiency of the singular arc to be optimal, are available.

The detailed treatment of a problem encountered in the structures literature follows. Allowance for the singularity is made for the first time.

L.3 A PROBLEM OF ARMAND IN ONE INDEPENDENT VARIABLE

L.3.1 Introduction and basic data. The foregoing comments and theory will now be applied to the work of Armand as an illustration of the occurrence of singular solutions and their treatment in structural design. The particular problem considered is Armand (1972, pp 26-79) but with the suggested state equations given on pp 13-14 and in appendix 1 of the same reference. Notice however that Armand's problems (his pages 80-122) are of a similar construction and hence singular in the same sense. It is shown in the present article that Armand's final solutions are nevertheless optimal despite not having recognised the singular condition.

The problem is reduced to a lumped parameter format for illustration purposes, and the maximum principle applied - Armand's original work employed a like minimum principle. The difference between the maximum and minimum principles is solely one of sign convention in defining the Hamiltonian and a related reversal of an inequality. (See §J.) The maximum principle agrees with the original Russian formulation of Pontryagin and coworkers.

A comparison with a nonsingular formulation and its solution is included (§L.3.3). Note that Armand solves the same problem (or two closely related versions of the same problem) for the distributed parameter case. Both are singular, but to different extents. When the equivalent lumped parameter problem is considered, the cause of the singularities remains in one case but disappears in the other. (The cause of the singularities in the first case results from satisfying the mathematics alone in formulating the system model without regard to physical meaning; this cause is independent of lumped or distributed parameter formats. The cause in the second case is inherent in system type II models where auxiliary controls exist.) The case where singularities remain in the lumped reduction is treated first in subarticle §L.3.2, the nonsingular case in §L.3.3. These two cases coincide (for the particular problem at hand) with structural models that have little physical meaning and complete physical meaning respectively.

It is remarked that the solution of the singular formulation of the problem does not require the use of such strong necessary conditions as given by the maximum principle. This point is taken up in a later subarticle (§L.3.4), where equivalent results are obtained using the calculus of variations. (The approaches - maximum principle and the calculus of variations - as stated before, are equivalent when there are no constraints on the control.)

Problem statement: A minimum mass design of a beam member, having elasticity only in shear, is sought. The fundamental frequency of a uniform reference beam is used as a constraint and the optimum distribution of material along the member is required. A constraint on the member geometry exists

Considering shear effects only, the differential equation of motion may be shown to be

$$(\ell.3.1) \qquad \frac{\partial}{\partial y}\left[h(y)\,\frac{\partial w(y,t)}{\partial y}\right] - \frac{\rho(y)}{G(y)}\,h(y)\,\frac{\partial^2 w(y,t)}{\partial t^2} = 0 \qquad y \in [0,L]$$

where $h(y)$ denotes the member thickness; $\rho(y)$ and $G(y)$ the material density and shear modulus respectively; and $w(y,t)$ the normal deflection.

The axis y has been chosen along the member axis.

For free vibrations of a shear beam of constant thickness and simply supported, it may be shown (in a similar manner to that given in the plate vibration illustration , section §G) that the modes and fundamental frequency of vibration are respectively

$$(\ell.3.2) \qquad W_m = A_m \sin \frac{m\pi y}{L} \qquad m = 1, 2, \ldots$$

$$(\ell.3.3) \qquad \omega_f = \omega_1 = \pi \sqrt{\frac{G}{\rho}\left(\frac{1}{L^2}\right)}$$

where $W_m = W_m(y)$ is the component of the displacement $w(y,t)$ dependent on the spatial coordinate y only.

Substituting the frequency constraint (ℓ.3.3), in equation (ℓ.3.1) gives

$$(\ell.3.4) \qquad \frac{d}{dy}\left(h \frac{dW}{dy}\right) + \frac{\rho}{G}\, \omega_f^2\, hW = 0$$

Boundary conditions are $W(y) = 0$ at $y = 0,L$.

The criterion is the mass of the member

$$(\ell.3.5) \qquad Q = \int_0^L \rho h(y)\, dy$$

In the following computations it is assumed that the total mass of the member involves two portions; a constant portion δ_2 which is non-structural, and a variable (structural) portion δ_1. The thickness may then be expressed as

$$(\ell.3.6) \qquad h(y) = \delta_1\, h^*(y) + \delta_2$$

where $\delta_1 + \delta_2 = 1$. Only the structural portion δ_1 may be controlled by the designer. (ℓ.3.5) becomes

$$Q = \rho \int_0^L h(y)\, dy = \rho\delta_1 \int_0^L h^*(y)\, dy + \rho\delta_2 L$$

or equivalently

$$(\ell.3.7) \qquad Q = \int_0^L h^*(y)\,dy$$

and the system equation, (ℓ.3.4), becomes

$$(\ell.3.8) \qquad \frac{d}{dy}\left(h^*\frac{dW}{dy}\right) + \frac{\rho}{G}\,\omega_f^2(\delta_1 h^* + \delta_2)W = 0$$

A constraint on the structural portion h^* may be expressed as

$$(\ell.3.9) \qquad h^* \geq h_0^*$$

where h_0^* is a minimum thickness requirement.

L.3.2 <u>Singular formulation and solution.</u> Equation (ℓ.3.8) may be expanded into the form

$$\frac{dh^*}{dy}\;\frac{dW}{dy} + h^*\frac{d^2W}{dy^2} + \frac{\rho}{G}\,\omega_f^2\;(\delta_1 h^* + \delta_2)W = 0$$

Introduce the state vector $\underset{\sim}{x}$, and control u. Set

$$\begin{bmatrix} x_1 \triangleq W \\ \\ x_2 \triangleq \dfrac{dW}{dy} \\ \\ x_3 \triangleq h^* \end{bmatrix}$$

$$u \triangleq \frac{dh^*}{dy}$$

Differentiating x_1, x_2 and x_3 with respect to y gives the system equations;

$$(\ell.3.10)\qquad \begin{bmatrix} \dfrac{dx_1}{dy} \\ \\ \dfrac{dx_2}{dy} \\ \\ \dfrac{dx_3}{dy} \end{bmatrix} = \begin{bmatrix} x_2 \\ \\ -\dfrac{1}{x_3}[ux_2 + \dfrac{\rho}{G}\,\omega_f^2\,(\delta_1 x_3 + \delta_2)x_1] \\ \\ u \end{bmatrix}$$

with state boundary conditions;

$$(\ell.3.10a)\qquad x_1(0) = 0\ ,\qquad x_2(^L/2)\quad x_3(^L/2) = 0$$

Symmetry has been invoked, so that only half the member need be considered.

The criterion expressed in terms of the newly introduced variables;

$$Q = \int_0^{L/2} x_3\ dy$$

The Hamiltonian is then defined as

$$H = -x_3 + \lambda_1 x_2 - \frac{\lambda_2}{x_3}[ux_2 + \frac{\rho}{G}\,\omega_f^2\,(\delta_1 x_3 + \delta_2)x_1]$$

$$+\lambda_3 u + \zeta(x_3 - h_0^*)$$

where $\zeta(y) \leq 0$ for $x_3 = h_0^*$

$\zeta(y) = 0$ for $x_3 \geq h_0^*$

H is seen to be linear in the control, and hence the maximum principle fails to choose the optimal control directly in this case. The necessary conditions for an extremum according to the maximum principle are;

(a) The control inequation in which the Hamiltonian is a maximum for the optimal control. Notice that since u appears linearly in H, $\hat{u}$ is not directly determined. Nevertheless a value for $\hat{u}$ will be found such that the coefficient of u in H

$$(\ell.3.11) \qquad \sigma(y) = \frac{-\lambda_2 x_2}{x_3} + \lambda_3$$

is maintained at zero over the interval in which u is unconstrained.

(b) the canonical equations as represented by (i) the system equations (ℓ.3.10)

$$(\ell.3.10) \qquad \frac{dx_i(y)}{dy} = \frac{\partial H}{\partial \lambda_i(y)} \qquad i = 1, 2, 3$$

with state boundary conditions (ℓ.3.10a); and (ii) the adjoint equations

$$(\ell.3.12) \qquad \frac{d\lambda_i(y)}{dy} = \frac{-\partial H}{\partial x_i(y)} \qquad i = 1, 2, 3$$

with natural boundary conditions and transversality conditions

$$(\ell.3.12a) \qquad \lambda_1(^L/_2) = 0, \quad \lambda_2(0) = 0, \quad \lambda_3(0) = 0$$

$$\lambda_2(^L/_2) = \Lambda x_3(^L/_2), \quad \lambda_3(^L/_2) = \Lambda x_2(^L/_2)$$

where Λ is a Lagrange multiplier (§F.3).

Equations (ℓ.3.12) may be expanded to give

$$(\ell.3.12) \qquad \begin{bmatrix} \dfrac{d\lambda_1}{dy} \\ \\ \dfrac{d\lambda_2}{dy} \\ \\ \dfrac{d\lambda_3}{dy} \end{bmatrix} = \begin{bmatrix} \dfrac{\lambda_2}{x_3}\left[\dfrac{\rho}{G}\,\omega_f^2\,(\delta_1 x_3 + \delta_2)\right] \\ \\ -\lambda_1 + \dfrac{\lambda_2 u}{x_3} \\ \\ 1 - \dfrac{\lambda_2}{x_3^2}\left[u x_2 + \dfrac{\delta_2 x_1 \rho \omega_f^2}{G}\right] - \zeta \end{bmatrix}$$

Had the control u been bounded, H would be a minimum on these control boundaries. However for u free, the control arcs are singular and must be such that

$$\sigma(y) = -\frac{\lambda_2 x_2}{x_3} + \lambda_3$$

is maintained at zero along a subinterval of $[0, {}^{L}/2]$. For the case $\zeta = 0$ and differentiating

$$\frac{d}{dy}\left(\frac{\partial H}{\partial u}\right) = \frac{d}{dy}\left(\frac{-\lambda_2 x_2}{x_3} + \lambda_3\right)$$

$$= \frac{d\lambda_3}{dy} - \frac{\lambda_2}{x_3}\frac{dx_2}{dy} + \frac{\lambda_2 x_2}{x_3^2}\frac{dx_3}{dy} - \frac{x_2}{x_3}\frac{d\lambda_2}{dy}$$

Substituting from (ℓ.3.10) and (ℓ.3.13)

$$= 1 + \frac{\lambda_2 k^2 x_1}{x_3} + \frac{x_2 \lambda_1}{x_3}$$

where $k^2 = \frac{\rho\omega_f^2\delta_1}{G}$. This is still independent of u explicitly.

Further differentiation of $\frac{\partial H}{\partial u}$ with respect to y is required. That is

$$\frac{d^2}{dy^2}\left(\frac{\partial H}{\partial u}\right) = \frac{d}{dy}\left(1 + \frac{\lambda_2 k^2 x_1}{x_3} + \frac{x_2\lambda_1}{x_3}\right)$$

$$= [x_2\lambda_2 - x_1\lambda_1]\,[k^2 + \frac{\ell^2}{2x_3}] - \frac{\lambda_1 u x_2}{x_3}$$

after substituting from (ℓ.3.10) and (ℓ.3.12). $\ell^2 = \frac{\rho\omega_f^2\delta_2}{G}$

This expression now contains u. Had the terms in u cancelled in this expression, still further differentiation of $\frac{\partial H}{\partial u}$ with respect to y, that is $\frac{d^3}{dy^3}\left(\frac{\partial H}{\partial u}\right)$, ... , would have been tried until an expression containing u was obtained. Rearranging terms, after setting to zero

$$(\ell.3.13) \qquad u = \left[\frac{\lambda_2}{\lambda_1} - \frac{x_1}{x_2}\right]\left[k^2 x_3 + \ell^2/2\right]$$

which is the control law applicable on singular arcs.

To ascertain whether this control is minimizing, the necessary condition (ℓ.2.5) is employed (κ = 2 in this case)

$$(\ell.3.14) \qquad (-1)^2 \frac{\partial}{\partial u}\left[\frac{d^2}{dy^2}\left(\frac{\partial H}{\partial u}\right)\right] = -\frac{\lambda_1 x_2}{x_3}$$

The problem may now be solved using the system and adjoint equations (ℓ.3.10) and (ℓ.3.12) with boundary conditions (ℓ.3.10a) and (ℓ.3.12a). The system and adjoint equations are related through the control given by (ℓ.3.13).

For a solution on the interval $[0, {}^{L}/2]$,(invoking symmetry), make the initial substitution (to be verified later);

$$x_1 = \frac{a}{k} \sinh[ky] \qquad a = \text{constant}$$

which by inspection satisfies the boundary condition on x_1. From $(\ell.3.10)^1$;

$$x_2 = a \cosh[ky]$$

These expressions for x_1 and x_2 may be substituted in $(\ell.3.10)^2$ leading to

$$x_3 = \frac{-u \cosh[ky]}{2k \sinh[ky]} - \frac{\ell^2}{2k^2}$$

With this expression for x_3, equation $(\ell.3.10)^3$ may be used to obtain an equation solely in terms of u, which has a general solution

$$u = \frac{bk \sinh[ky]}{\cosh^3[ky]} \qquad b = \text{constant}$$

Substituting for u in the expression for x_3, then

$$x_3 = \frac{-bk^2 - \ell^2\cosh^2[ky]}{2k^2 \cosh^2[ky]}$$

The adjoint equation $(\ell.3.12)^2$ may be reordered into the form

$$(\ell.3.12)^2 \qquad \frac{1}{\lambda_2}\frac{d\lambda_2}{dy} = -\frac{\lambda_1}{\lambda_2} + \frac{u}{x_3}$$

and the control equation $(\ell.3.14)$ into the form

$$\frac{\lambda_1}{\lambda_2} = \frac{x_2[k^2x_3 + \ell^2/2]}{ux_2 + x_1[k^2x_3 + \ell^2/2]}$$

This last expression for $\frac{\lambda_1}{\lambda_2}$ may be substituted into $(\ell.3.12)^2$ and replacing x_1, x_2, x_3 and u with their expressions from above, then an equation in λ_2 results;

$$\frac{1}{\lambda_2}\frac{d\lambda_2}{dy} = \frac{k\cosh[ky]}{\sinh[ky]} - \frac{2k\sinh[ky]}{\cosh[ky]}\left[\frac{1}{1+\dfrac{\ell^2\cosh^2[ky]}{bk^2}}\right]$$

Integrating

$$\lambda_2 = c\sinh[ky]\left[\frac{-\dfrac{bk^2}{\ell^2} - \cosh^2[ky]}{\cosh^2[ky]}\right]$$

where c is a constant of integration. This expression for λ_2 satisfies the boundary condition on λ_2; $\lambda_2(0) = 0$.

Substituting this value for λ_2 into $(\ell.3.12)^2$

$$\lambda_1 = \frac{\lambda_2 u}{x_3} - \frac{d\lambda_2}{dy}$$

$$= \frac{cbk^3}{\ell^2\cosh[ky]} + ck\cosh[ky]$$

To evaluate λ_3, use equation $(\ell.3.12)^3$

$$\frac{d\lambda_3}{dy} = 1 + \frac{4ak^3c}{\ell^2}\sinh^2[ky]$$

Integrating $\lambda_3 = y + \dfrac{2ack^2}{\ell^2}[\sinh[ky]\ \cosh[ky] - ky] + d$

The constant of integration d = 0 upon using the condition $\lambda_3(0) = 0$. Using the two transversality conditions on λ_2 and λ_3 at $y = {}^L/2$, namely

$$\lambda_2({}^L/2) = \Lambda x_3({}^L/2)$$

$$\lambda_3({}^L/2) = \Lambda x_2({}^L/2)$$

(equated through the Lagrange multiplier Λ) and the above estimates for λ_2, λ_3, x_2 and x_3, the constant c is evaluated as

$$c = \frac{\ell^2}{2k^3a}$$

The boundary condition $\lambda_1({}^L/2)$ implies

$$b = -\frac{\ell^2}{k^2}\cosh^2[{}^{kL}/2]$$

which also ensures the condition $x_2(0)\ x_3(0) = 0$ is satisfied.

The initial assumption for x_1 may now be verified as being correct using equation (ℓ.3.12)[1] for example. Equations (ℓ.3.10), (ℓ.3.12) and (ℓ.3.13) and boundary conditions (ℓ.3.10a) and (ℓ.3.12a) are satisfied and hence the substitution is valid.

To summarize the solution over $0 \leq y \leq {}^L/2$ for the unconstrained case ($\zeta = 0$);

$$x_1 = \frac{a}{k}\sinh[ky]$$

$$x_2 = a\cosh[ky]$$

$$x_3 = \frac{\ell^2}{2k^2}\left[\frac{\cosh^2[^{kL}/2] - \cosh^2[ky]}{\cosh^2[ky]}\right]$$

$$u = \frac{-\ell^2}{k^2}\left[\frac{k\ \sinh[ky]\ \cosh^2[^{kL}/2]}{\cosh^3[ky]}\right]$$

(ℓ.3.15)

$$\lambda_1 = -\frac{\ell^2}{2ak^2}\left[\frac{\cosh^2[\ ^{kL}/2] - \cosh^2[ky]}{\cosh[ky]}\right]$$

$$\lambda_2 = \frac{\ell^2}{2ak^3}\sinh[ky]\left[\frac{\cosh^2[^{kL}/2] - \cosh^2[ky]}{\cosh^2[ky]}\right]$$

$$\lambda_3 = \frac{1}{k}\sinh[ky]\ \cosh[ky]$$

By a symmetry argument, a similar solution applies over $^L/2 \leq y \leq L$. The constant a may be thought of as a modal amplitude factor. It only relates to the modal shape of W and not the geometry.

Substituting these values in (ℓ.3.14)

$$(-1)^2 \frac{\partial}{\partial u}\left[\frac{d^2}{dy^2}\left(\frac{\partial H}{\partial u}\right)\right] = -\frac{\lambda_1 x_2}{x_3}$$

$$= \cosh^2[ky]$$

$$> 0 \quad \text{always,}$$

and hence the control is minimizing as desired. Since no control constraints are present in this case, the solution will contain only one arc - the singular arc.

Staying with an overview of the problem, in an attempt to simplify the computations for the constrained case ($\zeta \leq 0$), it is apparent that the optimal control for the interval $[\ 0,\ ^L/2]$ will consist of at most two types of segments; segments off ($\zeta = 0$) and segments on ($\zeta \leq 0$) the geometry constraint (ℓ.3.9). Assuming a solution involving only one of each type of segment (per half member), the junction of the two segments will be a single point, y = y' say. Hence envisaging the solution of two subproblems;

(i) For the (singular) segment off the constraint, the solution is given by equations (ℓ.3.10), (ℓ.3.12) and (ℓ.3.13) with modified state and adjoint boundary conditions

$$x_1(0) = 0, \qquad x_3(y') = h_0^*$$

$$\lambda_2(0) = 0, \qquad \lambda_3(0) = 0$$

This is a related problem to that for which the solution is equations (ℓ.3.15) and only differs in the boundary conditions.

(ii) For the (nonsingular) segment on the constraint, x_3 is constant and equals h_0^*. It follows that the system equations may be solved directly. The boundary condition for this subproblem is $h_0^* x_2({}^L/2) = 0$

(iii) To complete the required number of boundary conditions needed to solve the present subproblems, continuity of the states x_1 and x_2 is invoked at the junction $y = y'$. x_3 is specified at y' and hence continuity conditions on x_3 are implicitly satisfied.

For the off-constraint (singular) segment, the form of x_1, x_2, x_3 and u follow as before. The constant b may be determined from the boundary condition $x_3(y') = h_0^*$, giving

$$b = -(2h_0^* + {}^{\ell^2}/k^2)\cosh^2[ky']$$

This determines the expressions for x_3 and u, from which λ_1, λ_2 and λ_3 may be calculated as before. To evaluate c, allow the constrained segment to vanish and apply the above transversality conditions on λ_2 and λ_3 at $y = {}^L/2$. The final form of the variables is as follows;

$$x_1 = \frac{a}{k}\sinh[ky]$$

$$x_2 = a\cosh[ky]$$

$$x_3 = \frac{-\ell^2}{2k^2} + (h_0^* + \frac{\ell^2}{2k^2})\,\frac{\cosh^2[ky']}{\cosh^2[ky]}$$

$$u = -2k(h_0^* + \frac{\ell^2}{2k^2}) \frac{\cosh^2[ky'] \sinh[ky]}{\cosh^3[ky]}$$

(ℓ.3.16)

$$\lambda_1 = -\frac{1}{a}\left[\frac{(h_0^* + \frac{\ell^2}{2k^2}) \cosh^2[ky'] - \frac{\ell^2}{2k^2}\cosh^2[ky]}{\cosh[ky]}\right]$$

$$\lambda_2 = \frac{1}{ak}\sinh[ky]\left[\frac{(h_0^* + \frac{\ell^2}{2k^2}) \cosh^2[ky'] - \frac{\ell^2}{2k^2}\cosh^2[ky]}{\cosh^2[ky]}\right]$$

$$\lambda_3 = \frac{1}{k}\sinh[ky] \cosh[ky]$$

The factor a will in general not be the same for the constrained and unconstrained cases although the usage and meaning is similar.

For the on-constraint (nonsingular) segment, $x_3 = h_0^*$ and equations (ℓ.3.10) combine to give the second order equation

$$\frac{d^2x_1}{dy^2} + \gamma^2 x_1 = 0 \qquad \text{where } \gamma^2 = k^2 + \frac{\ell^2}{h_0^*}$$

which has a general solution $x_1 = e_1 \sin(\gamma y) + e_2 \cos(\gamma y)$.

Using the condition $h_0^* x_2(L/2) = 0$, $e_2 = e_1 \frac{\cos \gamma L/2}{\sin \gamma L/2}$

That is $$x_1 = e_1[\sin(\gamma y) + \frac{\cos \gamma L/2}{\sin \gamma L/2}\cos(\gamma y)]$$

$$x_2 = e_1\gamma[\cos(\gamma y) - \frac{\cos \gamma L/2}{\sin \gamma L/2}\sin(\gamma y)]$$

Using the continuity conditions on the states x_1 and x_2 at y' gives

$$\frac{a}{k}\sinh[ky'] = e_1[\sin(\gamma y') + \frac{\cos \gamma L/2}{\sin \gamma L/2}\cos(\gamma y')]$$

$$a \cosh[ky'] = e_1\gamma[\cos(\gamma y') - \frac{\cos \gamma L/2}{\sin \gamma L/2}\cos(\gamma y')]$$

Eliminating e_1 from the last two equations produces a solution implicit in y', the junction position of the constrained and unconstrained segments;

(ℓ.3.17) $$\frac{k}{\gamma} = \tanh[ky'] \tan({}^{\gamma L}/2 - \gamma y')$$

and $$x_1 = \frac{a}{k}\sinh[ky'] \frac{[\sin(\gamma y) + \frac{\cos^{\gamma L}/2}{\sin^{\gamma L}/2}\cos(\gamma y)]}{[\sin(\gamma y') + \frac{\cos^{\gamma L}/2}{\sin^{\gamma L}/2}\cos(\gamma y')]}$$

(ℓ.3.18)

$$x_2 = \frac{\gamma a}{k}\sinh[ky'] \frac{[\cos(\gamma y) - \frac{\cos^{\gamma L}/2}{\sin^{\gamma L}/2}\sin(\gamma y)]}{[\sin(\gamma y') + \frac{\cos^{\gamma L}/2}{\sin^{\gamma L}/2}\cos(\gamma y')]}$$

$$x_3 = h_0^*$$

Summarizing the expressions for the optimal thickness;

$0 \leq y \leq y'$

$$h^* = -\frac{\ell^2}{2k^2} + (h_0^* + \frac{\ell^2}{2k^2})\frac{\cosh^2[ky']}{\cosh^2[ky]} \qquad \text{giving}$$

(ℓ.3.19)

$$h = \frac{\delta_2}{2} + (\delta_1 h_0^* + \frac{\delta_2}{2})\frac{\cosh^2[ky']}{\cosh^2[ky]}$$

$y' \leq y \leq {}^{L}/2$

$$h^* = h_0^* \qquad \text{giving}$$

(ℓ.3.20) $$h = \delta_1 h_0^* + \delta_2$$

where the junction location y' may be found from

(ℓ.3.17) $$\frac{k}{\gamma} = \tanh[ky'] \tan({}^{\gamma L}/2 - \gamma y')$$

These solutions can be shown to be equivalent to those obtained by Armand (1971, 1972) although he has not allowed for singularities in his computations.

L.3.3 A comparison with the solution to a nonsingular formulation.

The system equation as before,

$$(\ell.3.8) \qquad \frac{d}{dy}\left(h^*\frac{dW}{dy}\right) + \frac{\rho}{G}\,\omega_f^2(\delta_1 h^* + \delta_2)W = 0$$

may be reformulated by introducing the new state variables

$$(\ell.3.21) \qquad \begin{bmatrix} \xi_1 \hat{=} W \\ \\ \xi_2 \hat{=} h^*\dfrac{dW}{dy} \end{bmatrix}$$

and control $\mu \hat{=} h^*$. The components of the state vector may be interpreted as displacement and shear force, and the control as the geometry. Notice that these are different state and control to the previous article.

Differentiating the state variables ξ_1 and ξ_2 with respect to y

$$(\ell.3.22) \qquad \begin{bmatrix} \dfrac{d\xi_1}{dy} \\ \\ \dfrac{d\xi_2}{dy} \end{bmatrix} = \begin{bmatrix} \dfrac{\xi_2}{u} \\ \\ \dfrac{-\rho}{G}\,\omega_f^2(\delta_1\mu + \delta_2)\xi_1 \end{bmatrix}$$

Equations (ℓ.3.22) are the new system equations, with boundary conditions

$$(\ell.3.22a) \qquad \xi_1(0) = 0 \quad , \quad \xi_2(L/2) = 0$$

The criterion as in the previous article

$$Q = \int_0^{L/2} h^* dy$$

becomes

$$Q = \int_0^{L/2} \mu\, dy$$

The Hamiltonian may now be written as

$$(\ell.3.23) \qquad H = -\mu + \frac{\psi_1\xi_2}{\mu} + \psi_2\left(\frac{-\rho}{G}\,\omega_f^2(\delta_1\mu + \delta_2)\xi_1\right) + \zeta(\mu - h_0^*)$$

where $\zeta(y) \leq 0$ for $\mu = h_0^*$

$\zeta(y) = 0$ for $\mu \geq h_0^*$

For the optimal control

$$(\ell.3.24) \qquad \frac{\partial H}{\partial \mu} = 0 = -1 - \frac{\psi_1\xi_2}{\mu^2} - \psi_2\,\xi_1\,\frac{\rho}{G}\,\omega_f^2\delta_1 + \zeta$$

Recalling the symbol $k^2 = \frac{\rho}{G}\,\omega_f^2\delta_1$, then the optimal control is given by

$$(\ell.3.25) \qquad \mu^2 = \frac{-\psi_1\xi_2}{(1 + k^2\psi_2\xi_1 - \zeta)}$$

The adjoint equations become

$$(\ell.3.26) \qquad \begin{bmatrix} \dfrac{d\psi_1}{dy} \\ \\ \dfrac{d\psi_2}{dy} \end{bmatrix} = \begin{bmatrix} \psi_2\,\dfrac{\rho}{G}\,\omega_f^2(\delta_1\mu + \delta_2) \\ \\ \dfrac{-\psi_1}{\mu} \end{bmatrix}$$

with natural boundary conditions

$$(\ell.3.26a) \qquad \psi_1(L/2) = 0 \quad , \quad \psi_2(0) = 0$$

By inspection it may be seen that

$$(\ell.3.27) \qquad \psi_1 = -\frac{1}{A^2}\,\xi_2 \quad , \quad \psi_2 = \frac{1}{A^2}\,\xi_1$$

where A^2 is an undetermined constant, and may be thought of as a multiplier on the modal shapes of the beam. Note that this substitution is compatible with the boundary conditions also. By relating the state and adjoint variables in this way, the problem has now been reduced to the solution of a pair (either state or adjoint) of equations, in place of the four (state plus adjoint) equations previously. As the state variables have more direct physical significance, the solution will be given in terms of them. The optimality condition (ℓ.3.25) similarly reduces to

$$(\ell.3.28) \qquad \mu^2 = \frac{\xi_2^2}{(A^2 + k^2\xi_1^2 - A^2\zeta)}$$

(ℓ.3.28) implies a positive A^2 (and hence the reason for the original choice of this constant in squared form) for positive $\mu(0)$.

Consider the solution in the unconstrained case (ζ=0) firstly. Substituting the value of μ from (ℓ.3.28) into (ℓ.3.21) gives

$$(\ell.3.29) \qquad \begin{aligned} \frac{d\xi_1}{dy} &= (A^2 + k^2\xi_1^2)^{\frac{1}{2}} \\ \frac{d\xi_2}{dy} &= -\frac{\rho}{G}\,\omega_f^2 \left[\frac{\delta_1\xi_2}{(A^2 + k^2\xi_1^2)^{\frac{1}{2}}} + \delta_2\right]\xi_1 \end{aligned}$$

with boundary conditions $\xi_1(0) = 0$, $\xi_2(^L/2) = 0$ unchanged.

Equation $(\ell.3.29)^1$ contains ξ_1 alone and has the solution

$$(\ell.3.30)^1 \qquad \xi_1 = \frac{A}{k}\sinh[ky]$$

using the boundary condition on ξ_1. With this expression for ξ_1, $(\ell.3.29)^2$ becomes

$$\frac{d\xi_2}{dy} = -k\xi_2\tanh[ky] - \frac{\ell^2 A}{k}\sinh[ky]$$

where $\ell^2 = \frac{\rho}{G}\omega_f^2\delta_2$, and has the general solution

$$(\ell.3.31) \qquad \xi_2 = \left(\frac{1}{\cosh[ky]}\right)\left[\left(\frac{-\delta_2 A}{2\delta_1}\right)\cosh^2[ky] + B\right] \quad , \; B = \text{constant}$$

Applying the boundary condition $\xi_2(^L/2) = 0$, then

$$(\ell.3.30)^2 \qquad \xi_2 = \frac{\delta_2 A}{2\delta_1 \cosh[ky]} \left[\cosh^2[^{kL}/2] - \cosh^2[ky]\right]$$

It only remains to determine μ. From (ℓ.3.28)

$$(\ell.3.30)^3 \qquad \mu = \frac{\delta_2}{2\delta_1} \left[\frac{\cosh^2[^{kL}/2] - \cosh^2[ky]}{\cosh^2[ky]}\right]$$

Notice that μ is independent of the multiplier A.

Consider now the solution to the constrained problem ($\zeta \leq 0$). The optimal control will consist of two types of segments; segments off and on the constraint. Denoting the junction of the off and on segments by $y = y'$, then a valid solution will be continuous in the states ξ_1 and ξ_2 at y'. $\xi_1(0) = 0$ and $\xi_2(^L/2) = 0$ as before.

For the portion off the constraint, $0 \leq y \leq y'$, the solution follows closely that just given. In particular it was found that

$$(\ell.3.30)^1 \qquad \xi_1 = \frac{A}{k} \sinh[ky]$$

$$(\ell.3.31) \qquad \xi_2 = \frac{1}{\cosh[ky]}\left[B - \frac{\delta_2 A}{2\delta_1}\cosh^2[ky]\right]$$

Equation (ℓ.3.30)^1 satisfies the boundary condition on ξ_1, namely $\xi_1(0) = 0$. To evaluate the boundary condition needed to determine the constant B in (ℓ.3.31), consider the control equation, (ℓ.3.28). At y'

$$h_0^* = \frac{\xi_2}{(A^2 + k^2\xi_1^2)^{\frac{1}{2}}}$$

or

$$\xi_2(y') = h_0^*(A^2 + k^2\xi_1^2)^{\frac{1}{2}}$$

$$= h_0^* A \cosh[ky']$$

Applying this value to (ℓ.3.31), and after some elementary manipulations

$$\xi_2 = \frac{1}{\cosh[ky]} \left[\left(Ah_0^* + \frac{A\delta_2}{2\delta_1}\right) \cosh^2[ky'] - \frac{\delta_2 A}{2\delta_1} \cosh^2[ky]\right]$$

And using (ℓ.3.28)

$$\mu = \frac{-\delta_2}{2\delta_1} + \left(h_0^* + \frac{\delta_2}{2\delta_1}\right) \frac{\cosh^2[ky']}{\cosh^2[ky\ ]}$$

For the portion on the constraint, $y' \leq y \leq {}^L/2$, $\mu = h_0^*$ and equations (ℓ.3.22) may be simplified and combined to give

$$\frac{d^2\xi_1}{dy^2} + \gamma^2\xi_1 = 0 \qquad \text{where} \qquad \gamma^2 = \frac{\rho}{G}\,\omega_f^2\left(\delta_1 + \frac{\delta_2}{h_0^*}\right)$$

This is the familiar simple harmonic motion equation and has the general trigonometric solution

$$\xi_1 = C_1 \sin(\gamma y) + C_2 \cos(\gamma y)$$

and hence

$$\xi_2 = h_0^*[C_1\gamma \cos(\gamma y) - C_2\gamma \sin(\gamma y)]$$

using the boundary condition on ξ_2, namely $\xi_2({}^L/2) = 0$,

$$C_2 = C_1 \frac{\cos^{\gamma L}/2}{\sin^{\gamma L}/2}$$

Continuity conditions at y' will now be invoked on the states ξ_1 and ξ_2. Equating the values of ξ_1 and ξ_2 to the left and right of y', after substituting for C_2;

$$\frac{A}{k} \sinh[ky'] = C_1 [\sin(\gamma y') + \frac{\cos^{\gamma L}/2}{\sin^{\gamma L}/2} \cos(\gamma y')]$$

$$h_0^* A \cosh[ky'] = C_1\gamma h_0^* [\cos(\gamma y') - \frac{\cos^{\gamma L}/2}{\sin^{\gamma L}/2} \sin(\gamma y')]$$

From the first of these equations

$$C_1 = \frac{\frac{A}{k}\sinh[ky']}{[\sin(\gamma y') + \frac{\cos^{\gamma L}/2}{\sin^{\gamma L}/2}\cos(\gamma y')]}$$

Substituting for C_1 in the second of these equations, and after simplification

(ℓ.3.32) $$\frac{k}{\gamma} = \tanh[ky'] \tan(^{\gamma L}/2 - \gamma y')$$

which yields an implicit solution in y', the junction of the on-constraint and off-constraint segments.

The states ξ_1 and ξ_2 over the on-constraint region are found by back substituting the value of C_1 evaluated above in the expressions for ξ_1 and ξ_2;

$$\xi_1 = \frac{A}{k}\sinh[ky'] \frac{[\sin(\gamma y) + \frac{\cos^{\gamma L}/2}{\sin^{\gamma L}/2}\cos(\gamma y)]}{[\sin(\gamma y') + \frac{\cos^{\gamma L}/2}{\sin^{\gamma L}/2}\cos(\gamma y')]}$$

(ℓ.3.33)

$$\xi_2 = \frac{h_0^*\gamma A}{k}\sinh[ky'] \frac{[\cos(\gamma y) - \frac{\cos^{\gamma L}/2}{\sin^{\gamma L}/2}\sin(\gamma y)]}{[\sin(\gamma y') + \frac{\cos^{\gamma L}/2}{\sin^{\gamma L}/2}\cos(\gamma y')]}$$

In summary, the expressions for the optimal thickness are:

$0 \leq y \leq y'$

(ℓ.3.34) $$h^* = \frac{-\delta_2}{2\delta_1} + \left(h_0^* + \frac{\delta_2}{2\delta_1}\right)\frac{\cosh^2[ky']}{\cosh^2[ky\ ]}$$

giving

$$h = \delta_1 h^* + \delta_2$$

$$= \frac{\delta_2}{2} + \left(\delta_1 h_0^* + \frac{\delta_2}{2}\right)\frac{\cosh^2[ky']}{\cosh^2[ky\ ]}$$

$y' \leq y \leq {}^{L}/_{2}$

$$(\ell.3.35) \qquad h^* = h_0^*$$

giving

$$h = \delta_1 h_0^* + \delta_2$$

Symmetrical expressions exist for ${}^{L}/_{2} \leq y \leq L$.

The solutions for both the unconstrained and constrained problems are seen to be the same as their counterparts in the previous subarticle. Evidently the factors a (of §L.3.2) and A are equivalent.

L.3.4 The singular formulation and the calculus of variations.

It is remarked, by way of introduction, that the singular formulation of the present problem entails a free choice of control; that is no constraint has been defined. For such a problem, weaker necessary conditions than the maximum principle may have been used. In particular, the results of the calculus of variations of several dependent variables would have been sufficient; see for example Bliss (1946), Bolza (1931), Courant and Hilbert (1953), Elsgolc (1961) and Gelfand and Fomin (1963) among other standard works.

By reducing the problem to the calculus of variations format, the fundamental nature of singular solutions may be seen more clearly although the treatment may not be as convenient as with a maximum principle format. The similarities with the necessary conditions of the maximum principle for the unconstrained control case may be noted in passing.

The problem as outlined in subarticle §L.3.2, must first be converted into a form suitable for the application of the calculus of variations. Using a device of Berkovitz (1961), a new variable, x_4, is introduced with the properties

$$\frac{dx_4}{dy} \triangleq u \qquad \text{and} \qquad x_4(0) = 0$$

It is assumed that $\frac{dx_4}{dy}$ is free to take any value.

The problem may then be stated as; minimise

$$Q = \int_0^{L/2} x_3 dy$$

subject to

$$\begin{bmatrix} \frac{dx_1}{dy} \\ \\ \frac{dx_2}{dy} \\ \\ \frac{dx_3}{dy} \end{bmatrix} = \begin{bmatrix} x_2 \\ \\ \frac{-1}{x_3}\left\{\frac{dx_4}{dy} x_2 + \frac{\rho}{G}\omega_f^2(\delta_1 x_3 + \delta_2) x_1\right\} \\ \\ \frac{dx_4}{dy} \end{bmatrix}$$

The standard calculus of variations form of

$$(\ell.3.36) \qquad Q(\underline{x}) = \int_{y^L}^{y^R} L(y, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n) dy$$

where L is the Lagrangian, with side conditions

$$(\ell.3.37) \qquad \phi_k(y, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n) = 0 \qquad k = 1, \ldots, m \leq n$$

may be recognised. In the above, the superposed dot denotes differentiation with respect to y.

A standard problem in the calculus of variations is to determine the function $\underline{x}(y) = (x_1(y), \ldots, x_n(y))^T$ which minimizes the functional (ℓ.3.36) subject to the conditions (ℓ.3.37); the solution may be found in most treatises on the calculus of variations. To effect a solution, a function F (an augmented integrand) is defined as

$$F(y, \underline{x}, \underline{\dot{x}}, \underline{\lambda}) \triangleq L + \sum_{k=1}^{m} \lambda_k(y)\phi_k$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)^T$ are Lagrange (undetermined) multipliers. For the problem at hand, m = 3, n = 4 and

$$F = x_3 + \lambda_1\left[\frac{dx_1}{dy} - x_2\right] + \lambda_2\left[\frac{dx_2}{dy} + \frac{1}{x_3}\left\{\frac{dx_4}{dy}\ x_2 + \frac{\rho}{G}\omega_f^2(\delta_1 x_3+\delta_2)x_1\right\}\right]$$

$$+ \lambda_3\left[\frac{dx_3}{dy} - \frac{dx_4}{dy}\right]$$

F is linear in $\frac{dx_4}{dy}$ and $\frac{d^2F}{d\left[\frac{dx_4}{dy}\right]^2} = 0$

The calculus of variations requires a minimizing solution to satisfy (in addition to the side conditions (ℓ.3.37)) the conditions of Euler-Lagrange, Weierstrass and Legendre.(L and its partial derivatives with respect to its arguments up to the third order are taken to be continuous.)

The Euler-Lagrange equations

$$\frac{d}{dy}\left(\frac{\partial F}{\partial \dot{x}_i}\right) - \frac{\partial F}{\partial x_i} = 0 \qquad i = 1, \ldots , n$$

or

$$\sum_{j=1}^{n} \frac{\partial^2 F}{\partial \dot{x}_i\ \partial x_j}\dot{x}_j + \sum_{j=1}^{n} \frac{\partial^2 F}{\partial \dot{x}_i\ \partial \dot{x}_j}\ddot{x}_j + \frac{\partial^2 F}{\partial \dot{x}_i\ \partial y}$$

$$+ \sum_{k=1}^{m} \frac{\partial^2 F}{\partial \dot{x}_i\ \partial \lambda_k}\dot{\lambda}_k - \frac{\partial F}{\partial x_i} = 0$$

$$i = 1, \ldots , n$$

become, after rearranging

$$\frac{d\lambda_1}{dy} = \frac{\lambda_2}{x_3}\frac{\rho}{G}\omega_f^2(\delta_1 x_3 + \delta_2)$$

$$\frac{d\lambda_2}{dy} = -\lambda_1 + \frac{\lambda_2}{x_3}\frac{dx_4}{dy}$$

(ℓ.3.38)

$$\frac{d\lambda_3}{dy} = 1 - \frac{\lambda_2}{x_3^2}\left(x_2\frac{dx_4}{dy} + \frac{\rho}{G}\omega_f^2\delta_2 x_1\right)$$

$$\frac{d}{dy}\left(\lambda_3 - \frac{x_2\lambda_2}{x_3}\right) = 0$$

Associated with these equations are certain natural boundary and transversality conditions. If the above equations $(\ell.3.38)^{1,2,3}$ are compared with (ℓ.3.12) it is seen that the Lagrangian multipliers correspond with the adjoint variables of §L.3.2. (This correspondence breaks down however when the controls are restricted.) Berkovitz (1961) and Johnson (1965) show that $\lambda_3 - \frac{x_2\lambda_2}{x_3}$ is the only solution of $(\ell.3.38)^4$ which is compatible with the transversality conditions on $\underline{\lambda}$. This singular condition corresponds with the condition (ℓ.3.11) of subarticle §L.3.2.

The Weierstrass condition

$$E(y, \underline{x}, \dot{\underline{x}}, \dot{\underline{x}}^*, \underline{\lambda}) = F(y, \underline{x}, \dot{\underline{x}}^*, \underline{\lambda}) - F(y, \underline{x}, \dot{\underline{x}}, \underline{\lambda})$$

$$- \sum_{i=1}^{n} (\dot{x}_i^* - \dot{x}_i) \frac{\partial F(y, \underline{x}, \dot{\underline{x}}, \underline{\lambda})}{\partial \dot{x}_i}$$

for $\underline{x}(y) \neq \underline{x}^*(y)$, a neighbouring function. If $E \geq 0$, then $Q(\underline{x})$ is a minimum. By Taylor's theorem, expanding $F(y, \underline{x}, \dot{\underline{x}}^*, \underline{\lambda})$ about $F(y, \underline{x}, \dot{\underline{x}}, \underline{\lambda})$, the Weierstrass Excess Function becomes

$$E(y, \underline{x}, \dot{\underline{x}}, \dot{\underline{x}}^*, \underline{\lambda}) = \frac{(\dot{\underline{x}}^* - \dot{\underline{x}})^T}{2!} \frac{\partial^2 F(y, \underline{x}, \dot{\underline{x}}', \underline{\lambda})}{\partial \dot{x}_i \, \partial \dot{x}_j}$$

where $\dot{\underline{x}}'$ is a value between $\dot{\underline{x}}^*$ and $\dot{\underline{x}}$.

It may be shown (for example Elsgolc 1961) that the function E has a constant sign provided $\frac{\partial^2 F(y, \underline{x}, \dot{\underline{x}}', \underline{\lambda})}{\partial \dot{x}_i \, \partial \dot{x}_j}$ has.

Hence for the problem at hand $E = 0$.

The Legendre condition follows from the Weierstrass condition. For a minimum of the functional Q, the matrix with components $\frac{\partial^2 F}{\partial \dot{x}_i \, \partial \dot{x}_j}$ is required to be ≥ 0. $\frac{\partial^2 F}{\partial \dot{\underline{x}} \, \partial \dot{\underline{x}}}$ is the coefficient matrix of the $\ddot{\underline{x}}$ terms in the Euler-Lagrange equations and is a zero matrix for this problem.

Legendre's necessary condition is a weaker necessary condition than Weierstrass's condition, form which it may be derived.

Singularities thus yield Euler-Lagrange equations that are of a reduced order (coefficient of $\ddot{\underset{\sim}{x}}$ terms vanishes) and effectively are a 'degenerate' form of complete Euler-Lagrange equations. Also the Weierstrass and Legendre conditions are seen to be satisfied in the trivial sense and hence give no indication as to whether the function $\underset{\sim}{x}(y)$ minimizes Q or not. A more detailed discussion may be found in the excellent monograph of Johnson (1965).

L.4 DISCUSSION

A typical singular problem has been worked through and compared with a nonsingular formulation. In this example, the optimal solution to the nonsingular problem agreed with the optimal solution to the singular problem. There was a certain lack of physical meaning of the variables involved in the singular formulation but meaningful choices of the variables can also lead to singular formulations. The condition required to produce the singularity in the present case is a Hamiltonian linear in the control.

Two alternatives are open to the designer. Firstly he may show his awareness of the singular conditions and so formulate his problem to avoid them. No rigour is lost through such an approach although the efficiency of the solution process may be in doubt. Secondly he may recognise the singularity and adopt the corresponding solution process (that is the maximum principle with modifications) in order that an optimum solution is obtained.

Where bounds exist on the control, it was shown that a solution alternating between bounds may result. Such an optimal control is often referred to as a 'bang-bang' control. This form of control is common in aeronautical and electrical applications (for example full-thrust, no-thrust of a rocket, or the on-off positions of a switch). However it is unclear how a bang-bang type control would occur in structural applications (apart from restricting geometries, say, to discrete sizes). The closest structures may come to this case may be referred to as 'bang-singular-bang' where the switching function is zero over a finite interval and the control assumes its bounded values at the ends of this interval.

Previous structural design work has not considered the question of singularities. The reason for this appears to be an unawareness of the existence of the singular condition rather than uncertainty in the handling of the singular condition. This section, based on results developed in control systems theory, gives the necessary conditions that must be satisfied for optimality when singularities occur in the design formulation.

PART 3

STOCHASTIC DESIGN

§M DERIVATION OF CONDITIONS FOR OPTIMALITY

M.1 INTRODUCTION

M.1.1 Outline. The previous sections on design (§F to §L, part 2) assume that information about the system and system variables is completely specified by the values taken by their arguments. That is, the systems are deterministic. This assumption, of course, is a convenience, chosen for reasons of analytical tractability. Average effects only are considered in an attempt to simplify the concepts and computations. In the present and following two sections, an alternative mathematical model to determinism is employed for design calculations; namely stochasticism where uncertainty in a probabilistic sense is introduced. The system, system variables and parameters are allowed to assume a definite randomness.

In particular, this section derives a relationship for systematically determining the optimal control for stochastic systems. System parameters (including the loading) are assumed to be given with known probability distribution functions or alternatively to have known first and second moments. Variables are taken to have known statistical characteristics in accordance with the definition of stochastic systems given in article §C.2. The structure constitutive relationship and the optimality criterion are regarded as complete; that is they are known in a probabilistic sense. The behaviour of the system is predictable in a probabilistic sense. The expected value of the criterion is used as the measure of system suitability.

State-control modelling of the system equations is invoked as outlined in part 1. This leads to a natural assumption on the state as a Markov process which implies a weak form of dependency between states at successive values of the independent variable.

The derivation is for general nonlinear stochastic systems and criteria and hence covers a broad class of problems. For obvious conceptual reasons, lumped parameter problems only are entertained and hence for distributed parameter problems, some form of discretization procedure (as for example outlined in section §E) would be necessary to reduce it to a lumped form before computations were started. Restrictions may exist on the permitted range of any of the state or control variables. The extension to include further constraints (in particular reliability) is given in the following section (§N).

The value of the optimal control is shown to be determined uniquely from a knowledge of the state at any value of the independent variable and hence is nonrandom. It is argued that only nonrandom optimal controls exist and have meaning. See also Fel'dbaum (1960 - 1961) and Aoki (1965). Notice this distinction between the optimum problem and the analysis problem. In the latter case, the control is in general random, which together with given random parameters and boundary information on the states (for example end-state conditions, loading), produces a random state throughout the structure.

Section §O details an illustration of the usage of the optimality conditions derived in this section.

The previous sections in deterministic design (part 2) expressed the optimal control problem in terms of continuous variables and continuous admissible constraint regions. It will, however be found conceptually more convenient in handling stochastic design (part 3) if the variables are thought of as existing at a discrete number of locations and the admissible constraint sets thought of as involving a finite number of elements. The solution then entails a sequence of values in place of a function over the region of the structure. This discretization allows the optimization problem to reduce to a class of multistage decision processes (see §E) and provides useful results.

M.1.2 Background. In contrast to the deterministic portion of optimal control theory, the treatment given to stochastic problems has been of limited applicabiltiy and generally presented in an indefinite and sometimes obscure fashion. No definitive treatments of a general theory of stochastic optimal control exist. The large discrepancy between the states-of-the-art of the deterministic and stochastic cases may be partly attributable to the lack of direct coupling between the two cases and the greater diversity of stochastic problems. But it is felt that the inherent idea of probability is the greatest single cause for the lack of development of the theory in the stochastic case.

Related derivations to the one presented here may be found in Fel'dbaum (1960 - 1961, 1962) and Aoki (1965, 1967). Both Fel'dbaum and Aoki use statistical decision theory in conjunction with concepts related to dynamic programming for the discrete-time optimal control of systems with disturbances and incomplete information (learning systems). Markov random processes are used for the stochastic model of the system. Florentin (1962), Astrom (1965) and Stratonovich (1960) use similar reasoning for related adaptive control problems with noisy observations and both complete and incomplete state information. The foundations of the mathematical treatment of stochastic and adaptive systems have essentially been laid by Bellman (1957a, 1957b, 1958, 1961, 1962) and Bellman and Kalaba (1960); generally Markov properties are associated with the discrete form of dynamic programming to handle systems that are random in some sense. Florentin (1961) also uses the imbedding procedures of dynamic programming but in the continuous time sense for purely stochastic systems. The manner in which the continuous form is treated however is restrictive on the form of distributions allowed on the random variables. Wonham (1963) follows a similar path and couches the result in a Hamilton-Jacobi (stochastic) format. The state of stochasticism in optimal control is given in an historical summary and associated bibliography in Wonham (1963). See also Krasovskii and Lidskii (1961), Krasovskii (1960, 1962) and Fleming (1963) for related continuous time results of particular stochastic control problems.

While the basic idea for the present derivation has its origin in these works, they can be seen to be inapplicable to the structure's case in many important aspects. These discrepancies (essentially because the above works are treating different, though definitely related, problems) are accounted for in the present work while at the same time physical significance is given to the derivation. To the writer's knowledge, no equivalent work on continuous stochastic structural systems has been attempted to date. The reader is referred to the excellent work of Bolotin (1966, 1972) and Vorovich (1966) for an appreciation of the limited state of knowledge even for the far simpler analysis case. The coupling of state-control modelling and Markovian assumptions provides the basis for extending probabilistic arguments to the synthesis case. It is argued that the selection of an appropriate model has been the essential obstacle to the treatment of stochastic problems in the past.

M.2 BASIC PRELIMINARY INFORMATION

M.2.1 Notes and assumptions. Consider a system governed by a constitutive relationship of the form (§C.3),

$$\text{(m.2.1)} \qquad \frac{d\underline{x}(y)}{dy} = \underline{f}^*[\underline{x}(y),\ \underline{u}(y),\ y] \qquad \begin{array}{l} \underline{u}(y) \in U \\ y \in [y^L, y^R] \end{array}$$

where $\underline{x}(y)$ and $\underline{u}(y)$ are, respectively, the n-dimensional state vector process and the r-dimensional control vector process. The statistical characteristics of the processes are assumed known. $U \subset E^r$ is the admissible set of controls and is taken as a function of y only (see §D.2). $\underline{f}^*$ is in general a nonlinear n-vector function, variable with y.

End-state conditions are specified for $\underline{x}(y^L)$ and $\underline{x}(y^R)$ and may be deterministic or random.

It is assumed that the optimal control is chosen such that the integral criterion (§D.3)

$$\text{(m.2.2)} \qquad \tilde{Q} = \int_{y^L}^{y^R} G^*[\underline{x}(y),\ \underline{u}(y),\ y]\, dy$$

takes on a minimum value. G* is in general a given nonlinear function of its arguments (random) and hence itself is random. The equivalent deterministic measure will be taken as the expected value of $\tilde{Q}$.

Both functions, $\underline{f}^*$ and G*, are assumed to have a known form for all y.

To produce a conceptually simpler problem while avoiding the heavy rigour required in the continuous stochastic case, the parameter set Y (the closed interval $[y^L, y^R]$) is discretized while keeping the probability space continuous. The processes $\underline{x}(y)$ and $\underline{u}(y)$ now become random sequences and are completely defined by their 'finite dimensional distributions.'

Using the central difference expression at k(§E.2),

$$\text{(m.2.3)} \qquad \frac{d\underline{x}}{dy} = (\underline{x}^{k+1} - \underline{x}^{k})/\Delta$$

over $[y^L, y^R]$ partitioned into N subintervals $\Delta = (y^R - y^L)/N$, equation (m.2.1) becomes a vector difference equation

$$\text{(m.2.4)} \qquad \underline{x}^{k+1} = \underline{x}^{k} + \underline{f}[\underline{x}^{k}, \underline{u}^{k}, k] \qquad \underline{u}^{k} \in U^{k}$$

$$k = 0, 1, \ldots, N-1$$

where $\{\underline{u}^k \mathrel{\hat{=}} \underline{u}(k\Delta);\ k = 0, 1, \ldots, N-1\}$ and $\{\underline{x}^k = \underline{x}(k\Delta);\ k = 0, 1, \ldots, N-1\}$ are the random control and state sequences taking values at $y = k\Delta$. The behaviours of the discrete and the original continuous models are assumed to be similar as the subinterval Δ goes to zero. In equation (m.2.4), $\underline{f}[\underline{x}^k, \underline{u}^k, k] = \underline{f}^*[\underline{x}^k, \underline{u}^k, k]\Delta$ and U^k is the set of admissible controls in E^r. The model (m.2.4) may be interpreted as a sequence of transitions from the k'th to the (k+1)'th state, k = 0, 1, ... , N-1. With information only available on the states at discrete points, the control $\underline{u}^k$ is considered to be maintained constant during each subinterval and changed in a step manner at these points (figure M.2.1).

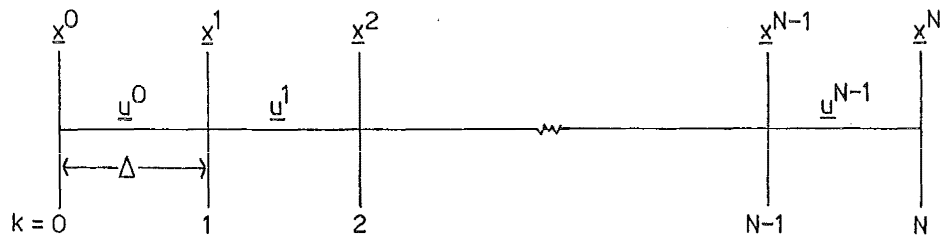

Figure M.2.1

The state is assumed to have Markov properties (§C.3), that is the state at $y = (k+1)\Delta$ depends only on the immediately previous state $\underline{x}^k$ and control $\underline{u}^k$.

Corresponding to the above, boundary conditions may be given a discrete-type notation. Conditions are specified at $y^L = 0\Delta$ and $y^R = N\Delta$.

($\underline{x}(y^L) \to \underline{x}^0$, $\underline{x}(y^R) \to \underline{x}^N$). Sufficient conditions have to be specified to completely define the model at any position $y \in [y^L, y^R]$.

The integral criterion, expression (m.2.2), to sufficient accuracy for small subinterval size Δ, is replaced by the summation

$$\tilde{Q} = \sum_{k=0}^{N-1} G^k[\underline{x}^k, \underline{u}^k, k]$$

where $G^k[\underline{x}^k, \underline{u}^k, k] = G^*[\underline{x}^k, \underline{u}^k, k]\Delta$

Denoting $M\{\cdot\}$ as the expectation operator, the deterministic measure of the system may be written (§D.3)

$$Q = M\{\tilde{Q}\} = M\{\sum_{k=0}^{N-1} G^k[\underline{x}^k, \underline{u}^k, k]\} \tag{m.2.5}$$

For completeness a criterion, which is a function of the right hand end-state alone (§D.3), will be appended to the above integral criterion and will be carried along in the same derivation manipulations. No special techniques are required to handle this additional information. If it is assumed that the probabilistic representation of this end-state criterion is of the form, say, $g[\underline{x}^N]$, where g is a general nonlinear function of $\underline{x}^N$ only, then the complete criterion is

$$Q = M\{\sum_{k=0}^{N-1} G^k[\underline{x}^k, \underline{u}^k, k] + g[\underline{x}^N]\} \tag{m.2.6}$$

Equation (m.2.5) contains a left hand end-state criterion as a special case and hence need not be mentioned further.

<u>M.2.2 Problem statement.</u> With the discretization complete, the problem may now be stated concisely as: To choose the controls $\underline{u}^k \in U^k$, $k = 0, 1, \dots, N-1$, such that

$$Q = M\{\sum_{k=0}^{N-1} G^k[\underline{x}^k, \underline{u}^k, k] + g[\underline{x}^N]\}$$

is minimised for a system behaving according to

$$\underline{x}^{k+1} = \underline{x}^{k} + \underline{f}[\underline{x}^{k}, \underline{u}^{k}, k] \qquad k = 0, 1, \ldots, N-1$$

with certain probabilistic end-state conditions. There may also exist certain restrictions on the range of the state variables $\underline{x}^k$.

M.3 DERIVATION OF THE CONDITIONS.

To effect a solution, assume firstly that only one interval between $y = (N-1)\Delta$ and $y = N\Delta$, is involved. It is assumed that the state $\underline{x}^{N-1}$ is known and it only remains to evaluate the $\underline{u}^{N-1}$ such that the expected value of the criterion over this interval is minimized. That is the problem is to minimize with respect to $\underline{u}^{N-1}$:

$$Q_{N-1} = M\{G^{N-1}(\underline{x}^{N-1}, \underline{u}^{N-1}, N-1) + g(\underline{x}^{N})\}$$

From equation (m.2.4) this becomes

$$Q_{N-1} = M\{G^{N-1}(\underline{x}^{N-1}, \underline{u}^{N-1}, N-1) + g(\underline{x}^{N-1} + \underline{f}[\underline{x}^{N-1}, \underline{u}^{N-1}, N-1])\}$$

$$= \int\{G^{N-1}(\underline{x}^{N-1}, \underline{u}^{N-1}, N-1) + g(\underline{x}^{N-1} + \underline{f}[\underline{x}^{N-1}, \underline{u}^{N-1}, N-1])\} \quad \text{(m.3.1)}$$

$$p(\underline{x}^{N-1}, \underline{u}^{N-1})\, d(\underline{x}^{N-1}, \underline{u}^{N-1})$$

The subscripted (N-1), embellishing Q, indicates the interval $(N-1)\Delta \leq y \leq N\Delta$. Equation (m.3.1) implies that both the state and the control are random quantities. G^{N-1} is a function of only the control $\underline{u}^{N-1} \in U^{N-1}$ and hence may be minimized with respect to this vector, giving $\underline{u}^{N-1}$ as a function of $\underline{x}^{N-1}$. The minimizing value of $\underline{u}^{N-1}$ is the optimal control for this interval starting with the state $\underline{x}^{N-1}$. Denoting $\hat{Q}_{N-1}$ as the minimum value of the criterion for this interval, $\hat{Q}_{N-1}$ will be a function of $\underline{x}^{N-1}$ only. Thus setting

$$c^{N-1} \triangleq p(\underline{u}^{N-1}) \qquad \underline{u}^{N-1} \in U^{N-1} \quad \text{(m.3.2)}$$

then

$$\hat{Q}_{N-1}(\underline{x}^{N-1}, N-1) \triangleq \min_{c^{N-1}} M\{G^{N-1}(\underline{x}^{N-1}, \underline{u}^{N-1}, N-1)$$

$$+ g(\underline{x}^{N-1} + \underline{f}[\underline{x}^{N-1}, \underline{u}^{N-1}, N-1])\}$$

$$= \min_{c^{N-1}} \int\{G^{N-1}(\underline{x}^{N-1}, \underline{u}^{N-1}, N-1) \tag{m.3.3}$$

$$+ g(\underline{x}^{N-1} + \underline{f}[\underline{x}^{N-1}, \underline{u}^{N-1}, N-1])\}$$

$$p(\underline{x}^{N-1}, \underline{u}^{N-1})\, d(\underline{x}^{N-1}, \underline{u}^{N-1})$$

All minimizations are understood to be subject to any restrictions or constraints attached to the problem.

Consider now that the two intervals for $(N-2)\Delta \leq y \leq N\Delta$, are involved. The assumption is made, in a similar manner to that pertaining to the one-interval-solution, that the state $\underline{x}^{N-2}$ is known. The value of the criterion for the two-interval-solution is the sum of the expectations for the two intervals.

$$Q_{N-2} = M\{G^{N-2}(\underline{x}^{N-2}, \underline{u}^{N-2}, N-2) + G^{N-1}(\underline{x}^{N-1}, \underline{u}^{N-1}, N-1)$$

$$+ g(\underline{x}^{N-1} + \underline{f}[\underline{x}^{N-1}, \underline{u}^{N-1}, N-1])\}$$

$$= M\{G^{N-2} + M[G^{N-1} + g|\underline{x}^{N-2}, \underline{u}^{N-2}]\}$$

$$= M\{G^{N-2} + \int\{G^{N-1}(\underline{x}^{N-1}, \underline{u}^{N-1}, N-1) \tag{m.3.4}$$

$$+ g(\underline{x}^{N-1} + \underline{f}[\underline{x}^{N-1}, \underline{u}^{N-1}, N-1])\}$$

$$p(\underline{x}^{N-1}, \underline{u}^{N-1} | \underline{x}^{N-2}, \underline{u}^{N-2})\, d(\underline{x}^{N-1}, \underline{u}^{N-1})\}$$

The probability density may be rewritten using an extended form of the definition of the conditional density function;

$$p(\underline{x}^{N-1}, \underline{u}^{N-1} | \underline{x}^{N-2}, \underline{u}^{N-2}) = p(\underline{u}^{N-1} | \underline{x}^{N-2}, \underline{u}^{N-2})\; p(\underline{x}^{N-1} | \underline{x}^{N-2}, \underline{u}^{N-1})$$

$$= p(\underline{u}^{N-1} | \underline{x}^{N-2}, \underline{u}^{N-2})\; p(\underline{x}^{N-1} | \underline{x}^{N-2}, \underline{u}^{N-2})$$

in which the Markov property of the state has been invoked in the last line. The second distribution may be obtained directly from the system equations.

If the optimal control is used in the last interval, then Q_{N-2} is minimized if $M\{G^{N-2} + \hat{Q}_{N-1}(\underline{x}^{N-1}, N-1) | \underline{x}^{N-2}, \underline{u}^{N-2}\}$ is minimized with respect to $\underline{u}^{N-2}$. The notation $\hat{Q}_{N-1}(\underline{x}^{N-1}, N-1 | \underline{x}^{N-2}, \underline{u}^{N-2})$ is used to denote the conditioning of the state $\underline{x}^{N-1}$ on the previous state $\underline{x}^{N-2}$ and control $\underline{u}^{N-2}$. Define in an analogous manner to equations (m.3.2) and (m.3.3);

$$\text{(m.3.5)} \qquad \begin{aligned} c^{N-2} &\triangleq p(\underline{u}^{N-2}) & \underline{u}^{N-2} &\in U^{N-2} \\ c^{N-1} &\triangleq p(\underline{u}^{N-1} | \underline{x}^{N-2}, \underline{u}^{N-2}) & \underline{u}^{N-1} &\in U^{N-1} \end{aligned}$$

Then

$$\hat{Q}_{N-2}(\underline{x}^{N-2}, N-2) = \min_{c^{N-2}} M\{G^{N-2} + \hat{Q}_{N-1}(\underline{x}^{N-1}, N-1 | \underline{x}^{N-2}, \underline{u}^{N-2})\}$$

$$\text{(m.3.6)} \qquad = \min_{c^{N-2}} \{M[G^{N-2}] + \int \hat{Q}_{N-1}(\underline{x}^{N-1}, N-1)$$

$$p(\underline{x}^{N-1} | \underline{x}^{N-2}, \underline{u}^{N-2}) d\underline{x}^{N-1}\}$$

Notice that the second term has already been found and in fact represents a conditional form of $\hat{Q}_{N-1}(\underline{x}^{N-1}, N-1)$ for the one-interval-solution with initial state $\underline{x}^{N-1}$. Substituting for $\underline{x}^{N-1}$ from the system equations, it is readily seen that equation (m.3.6) is a function of only the control $\underline{u}^{N-2}$. The minimization may be carried out to yield $\hat{Q}_{N-2}$ as a function of $\underline{x}^{N-2}$.

The above procedure may be repeated for any number of intervals j, $1 \leq j \leq N$. By similar reasoning

$$
\begin{aligned}
&c^{N-j} \triangleq p(\underline{u}^{N-j}) && \underline{u}^{N-j} \in U^{N-j} \\
&c^{N-j+1} \triangleq p(\underline{u}^{N-j+1} | \underline{x}^{N-j}, \underline{u}^{N-j}) && \underline{u}^{N-j+1} \in U^{N-j+1} \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&c^{N-1} \triangleq p(\underline{u}^{N-1} | \underline{x}^{N-2}, \underline{u}^{N-2}) && \underline{u}^{N-1} \in U^{N-1}
\end{aligned}
\qquad \text{(m.3.7)}
$$

and

$$
\begin{aligned}
\hat{Q}_{N-j}(\underline{x}^{N-j}, N-j) = \min_{c^{N-j}} \{ M[G^{N-j}] & \\
+ \int \hat{Q}_{N-j+1}(\underline{x}^{N-j+1}, N-j+1) & \\
p(\underline{x}^{N-j+1} | \underline{x}^{N-j}, \underline{u}^{N-j}) d\underline{x}^{N-j+1} \} &
\end{aligned}
\qquad \text{(m.3.8)}
$$

where $\hat{Q}_{N-j}(\underline{x}^{N-j}, N-j)$ is the minimum of the sum of the j-interval contributions to the criterion. $\underline{x}^{N-j+1}$ is related to $\underline{x}^{N-j}$ through the system equations. The minimization is over only one control $\underline{u}^{N-j}$.

Equations (m.3.8) and (m.3.3) are the general equations determining the optimum solution to the design problem, and although no general analytic method exists for their solution, systematic procedures for their solution may be worked out and can lead to analytic solutions.

The solution computations may be carried out, successively computing $\{\underline{u}^{N-j};\ j = 1, \ldots, N\}$, which minimizes $\{Q_{N-j};\ j = 1, \ldots, N\}$ conditioned on the preceding value of the state and control. The pairs $(\hat{\underline{u}}^{N-j}, \hat{Q}_{N-j})$ as functions of the characteristics (for example moments or the probability law) of $\underline{x}^{N-j}$ are calculated and stored in 'memory' for all j terminating with $(\hat{\underline{u}}^{0}, \hat{Q}_{0})$. $\hat{Q}_0$ is the 'cost' of control of the system while the sequence $\{\hat{\underline{u}}^{i};\ i = 0, 1, \ldots, N-1\}$ constitutes the optimal control sequence. The states may be evaluated from the system equations, equations (m.2.4) (where the control variables are now known) and the end-state conditions.

Three situations may arise in considering the end-state conditions:

(a) For end-state conditions specified solely at $y = y^L$, the conditions enter the computations in the last stage.

(b) For end-state conditions specified solely at $y = y^R$, the conditions enter the computations in the first stage; that is, in the determination of $\hat{Q}_{N-1}$ and $\hat{u}^{N-1}$. The value of $\hat{u}^{N-1}$ is chosen only in order that the state conditions at y^R are attained and not with regard to optimality. In this sense $\hat{Q}_{N-1}$ and $\hat{u}^{N-1}$ are determined automatically without any extremisation procedure. Having determined $\hat{Q}_{N-1}$ and $\hat{u}^{N-1}$, the remaining pairs $(\hat{u}^{N-j}, \hat{Q}_{N-j};\ j = 2, 3, \ldots N)$ are determined as for (a) without further reference to the end-state conditions.

(c) For 'mixed' end-state conditions, namely state coordinates specified at both y^L and y^R, the procedure is a combination of (a) and (b) above.

The above outline of the solution computations will apply where an analytical or hand solution is sought. For a fully systematic numerical solution the procedure is slightly different and is outlined in the following section (§N).

A note on the choice of optimal controls: The minimization operations in the above are performed with respect to c^{N-j}, the conditional probability densities of the controls $\underline{u}^{N-j}$. However after any j intervals, $\hat{\underline{u}}^{N-j}$ is uniquely determined for any given state $\underline{x}^{N-j}$ and hence only deterministic controls need be considered for optimal controls. The foregoing expressions consequently simplify with this conversion.

Expression (m.3.8) for a j-stage first order Markov process becomes

$$\hat{Q}_{n-j}(\underline{x}^{N-j}, N-j) = \min_{\underline{u}^{N-j} \in U^{N-j}} \{\int G^{N-j}(\underline{x}^{N-j}, \underline{u}^{N-j}, N-j)\, p(\underline{x}^{N-j})\, d\underline{x}^{N-j} + \int \hat{Q}_{N-j+1}(\underline{x}^{N-j+1}, N-j+1)\, p(\underline{x}^{N-j+1} | \underline{x}^{N-j}, \underline{u}^{N-j})\, d\underline{x}^{N-j+1}\} \quad \text{(m.3.9)}$$

or in terms of expectations

$$\hat{Q}_{N-j}(\underline{x}^{N-j}, N-j) = \min_{\underline{u}^{N-j} \in U^{N-j}} \{M[G^{N-j}(\underline{x}^{N-j}, \underline{u}^{N-j}, N-j)] + M[\hat{Q}_{N-j+1}(\underline{x}^{N-j+1}, N-j+1) | \underline{x}^{N-j}, \underline{u}^{N-j}]\} \quad \text{(m.3.10)}$$

This is in fact the functional recurrence relationship obtained by applying Bellman's principle of optimality (Bellman 1957) to

$$\hat{Q}_{N-j}(\underline{x}^{N-j}, N-j) = \min_{\substack{\underline{u}^{N-j} \in U^{N-j} \\ \cdot\ \cdot\ \cdot\ \cdot \\ \underline{u}^{N-1} \in U^{N-1}}} M\{\sum_{i=1}^{j} G^{N-i}(\underline{x}^{N-i}, \underline{u}^{N-i}, N-i) | \underline{x}^{N-i-1}, \underline{u}^{N-i-1}\}$$

with deterministic optimal controls and a conditioning that is only applicable for states following $\underline{x}^{N-j}$. Bellman's dynamic programming, being based on the principle of optimality, is essentially the repeated application of this recurrence relationship between successive system transitions.

M.4 DISCUSSION

The procedure for handling continuous stochastic systems was to introduce corresponding discrete systems which were taken to approximate to the original continuous form as the discretization interval approaches zero. The reason for employing some form of discretization should be evident. Notation-wise and concept-wise, the derivation remained quite tractable. However on going to the continuous case, a 'feel' for the problem is soon lost; together with the increased rigour required, this prohibits the development of a useful set of optimality conditions. Clearly the discretization intervals need not be equal in magnitude but may be varied to suit the problem. The intervals, Δ, in such a case would be superscripted to coincide with the relevant stage.

The foregoing derivation could equally well have been given in a forward manner (that is starting at the first interval) which would be entirely equivalent in principle. Alternatively, the derivation may have been avoided (though at the expense of losing insight into the problem) by using the imbedding procedure of dynamic programming, assuming the correct use of the principle of optimality. (In the present case there exist certain qualifying provisions, provisions which only become apparent from the verbal arguments employed above.) However the method presented does not rely on an explicit statement of the principle of optimality and hence holds far more appeal from an engineering or intuitive viewpoint.

The development of a variational optimal stochastic theory along the more classical lines of sections §F, §H and §J (see for example Pontryagin et al (1962), Kolmogorov et al (1962), Andreyev (1969)) appears very difficult. The present derivation will answer many questions concerning optimum stochastic structural design. However for a more complete set of answers, a complementary variational approach would appear desirable. The present work will however enable a designer to obtain a better feel for the random behaviour of structures, and design under random conditions.

The generality of the derivation is emphasized. It presupposes a knowledge only of certain probabilistic characteristics, such as density functions, for the variables concerned but makes no a priori assumption as to the exact nature of these characteristics. Obviously in different situations the characteristics will be different yet the derivation remains the same. The derivation is for general nonlinear, y-variant probabilistic systems and criteria. The extension to include multiple loadings is apparent and conceptually involves no new ideas.

Deficiencies in the present approach centre on the treatment of the end-state conditions and the solution of the optimal condition or recurrence relation. First, end-state conditions are not systematically treated. Logical reasoning may be used in many cases to overcome this, and although no rigour is lost, the efficiencey of the solution procedure remains in doubt. Second, although the approach provides an elegant treatment of stochastic structures and hence its relevance is assured, there may be limitations on its practical application arising out of difficulties that may be encountered in the solution of the recurrence relation. This point is taken up again in the following two sections. However it is emphasized that the problem of the optimum stochastic design of structures is a very difficult problem, as evidenced by the fact that nothing equivalent has been previously available in the literature, and consequently difficulties are to be expected irrespective of the type of solution procedure that evolves. A solution in the present work has essentially been made possible by the use of state modelling in conjunction with Markovian assumptions. Appropriate mathematical models have never before been available in the structures literature.

It is emphasized that the iterative nature of the equations ideally suits them for digital computation should a closed form solution not be available. It follows that non-analytic systems, criteria, and constraints may be dealt with using numerical computation procedures exclusively. Non-analyticity is present, for example, when restrictions exist on the permissible range of any of the variables $\underline{x}$ or $\underline{u}$. When treated numerically the computations are, in this case, simplified over the unconstrained numerical solution owing to the number of reduced alternatives that have to be considered. Another form of non-analyticity, and one that is treated in the following section, is that exhibited by reliability constraints. The complete numerical treatment of the present stochastic optimality conditions and the conditions with added non-analytic constraints is also given in the following two sections.

The significance of the conditions for optimality may be most easily clarified by an illustration and this may be found in section §0.

§N THE EXTENSION OF THE OPTIMALITY CONDITIONS OF §M TO INCLUDE RELIABILITY CONSTRAINTS

N.1 INTRODUCTION

N.1.1 Outline. Section §M derived a set of stochastic conditions for optimality which were applicable for the more usual forms of constraints on the state and control such as restrictions on the permissible ranges of any of the state or control variables. More complicated types of constraints restricting functions of both the state and control to permissible regions can however be included into the conditions of §M by suitably enlarging the arguments used. This form of constraint includes, as a special case, constraints on reliability and in general causes an increase in the dimensionality of the equations. The approach to the solution, however, remains fundamentally the same as in §M.

The solution in §N.3 is given in two parts. Subarticle §N.3.1 establishes the relevant reliability data in a form suitable for inclusion in the basic optimality conditions of §M. Subarticle §N.3.2 then shows the method for incorporating the reliability constraint into the basic formulation. A numerical procedure is then outlined for the solution in particular cases.

N.1.2 Background. The reliability problem is well defined in the literature for structural elements whose behaviour is described by a single entity (see for example Turkstra 1970, Tichy and Vorlicek 1972) but equivalent reliability derivations for structures with a continuously space-varying state are unknown. (Note however, reliability in a time sense has attracted much attention; see for example the various approaches of Bolotin 1972, Shinozuka 1964, Bogdanoff and Kozin 1961.) The extension to this case is given in appendix two, and underlines the fundamental nature of state concepts.

The optimization problem presented here is based on related work for discrete member structures by Kalaba (1962) and Khachaturian and Haider (1966) who use pseudo-sequential solution procedures on trusses despite the non sequential nature of this class of structure. Discussion on general global state-control constraints, though in neither a probabilistic nor reliability sense, may be found in Bellman (1961), Nemhauser (1966), and Boudarel et al (1971) among others.

N.2 THE EXTENDED PROBLEM.

N.2.1 Statement of the problem. The extended problem may be stated as: To determine the control such that the overall probability of failure of the structure will not exceed a given value, P_F say, while extremising some design criterion. Other constraints (for example on the control) will generally be present concurrently.

As in section §M on probabilistic optimization, the criterion will be taken to be (expression (m.2.5))

$$Q = M \{ \sum_{k=0}^{N-1} G^k [\underline{x}^k, \underline{u}^k, k] \}$$

The expected value is used for the deterministic measure of system suitability, since G^k has random arguments and hence will be a random quantity itself. The form of the criterion is left open in the derivation and includes performance, structural weight, cost and others as special cases.

This is a different optimization problem as Khachaturian and Haider (1966) note to the allocation of equal reliabilities to each of the structure component-parts. The problem in this case is discussed at a lower subsystem level and in general will not lead to an absolute extremum for the total structure but only one close to this extremum.

In essence, the problem solution requires a control policy and a reliability policy to be assigned throughout the structure so as to ensure an optimum structure of bounded reliability.

N.2.2 Assumptions. Assumptions remain the same as in section §M. In particular, the system parameters are assumed probabilistic and the state is again taken as a Markov process. No a priori decision as to the form of probability laws (for example normal, gamma, ...) for the problem variables is made. Computations are performed for general laws whereby specializing for particular problems may be undertaken in a routine manner. That is, the probability laws assumed are secondary to the arguments advanced.

Reliability is used in the sense outlined in appendix two. Notice the usage of the terminology 'state' in a reliability context. 'Failure' is used in the sense of exceeding a certain limit state.

N.3 SOLUTION.

N.3.1 Reliability data. For the system model of article §M.2,

$$\underline{x}^{k+1} = \underline{x}^k + \underline{f}[\underline{x}^k, \underline{u}^k, k] \qquad \underline{u}^k \in U^k, \quad k = 0, 1, \ldots, N-1 \tag{n.3.1}$$

defined at discrete points over the interval $[y^L, y^R]$, $\{\underline{u}^k = \underline{u}(k\Delta);\ k = 0, 1, \ldots, N-1\}$ and $\{\underline{x}^k = \underline{x}(k\Delta);\ k = 0, 1, \ldots, N\}$ are random sequences and take values at $y = k\Delta$. $\underline{x}^k$ is the n-dimensional state vector, $\underline{u}^k$ is the r-dimensional control vector and $\underline{f}$ is a real nonlinear n-vector function. $\underline{x}^k$ in (n.3.1) is assumed to have the properties of a Markov process. End-state conditions may be random or nonrandom.

Associated with each discretization point i, there exists a probability of failure as derived in appendix two. The system is assumed to fail when the state intersects the stochastic limiting surface ∂S. The probability of failure is then

$$\eta^i = [\iint p(\underline{r})\ p(\underline{x})\ d\underline{r}d\underline{x}]^i \qquad i = 0, 1, \ldots, N$$

where the integration is performed over a domain in E^{2n}. $\underline{r}$ denotes the structure 'resistance', and $\underline{x}$ denotes the structure state. (See note on terminology in appendix two.)

The assumption is made here in correspondence with a 'series connection type' reliability that failure at any position suggests failure of the whole structure.

A relationship, similar to the system equations (n.3.1) connecting the states at successive discretization points, is required between the probabilities of failure at successive points of the system. Without loss of generality, the probabilities of failure of the system will be computed in a backward sense, similar to the manner in which the criterion was evaluated in section §M. Then:

Probability of no failure to point (N-j)

= (probability of no failure to point (N-j+1))

$\cap$ (probability of no failure at (N-j)'th point)

That is $\quad [1-\rho^{N-j}] = [1-\rho^{N-j+1}]\,[1-\eta^{N-j}]$

which gives

$$\rho^{N-j+1} = \frac{[\rho^{N-j} - \eta^{N-j}]}{[1-\eta^{N-j}]} \tag{n.3.2}$$

where for general j, (j = 0, 1, ... , N), $\rho^{N-j} \simeq \sum_{i=N-j}^{N} \eta^{i}$ from appendix two.

Using an equivalent approximation to that involved in the last line, for η^{i} small

$$\rho^{N-j+1} \simeq \rho^{N-j} - \eta^{N-j} \tag{n.3.3}$$

N.3.2 Augmentation of the dimensionality. It will be recalled from section §M, equation (m.3.9), that the fundamental recurrence relationship for probabilistic optimization assuming a Markov process for the state was of the form

$$\begin{aligned}\hat{Q}_{N-j}(\underline{x}^{N-j},N-j) = \min_{\underline{u}^{N-j} \in U^{N-j}} \{ &\int G^{N-j}(\underline{x}^{N-j}, \underline{u}^{N-j}, N-j)\,p(\underline{x}^{N-j})\,d\underline{x}^{N-j} \\ &+ \int \hat{Q}_{N-j+1}(\underline{x}^{N-j+1}, N-j+1)\,p(\underline{x}^{N-j+1}|\underline{x}^{N-j}, \underline{u}^{N-j}) \\ &d\underline{x}^{N-j+1}\}\end{aligned} \tag{n.3.4}$$

where $\hat{Q}_{N-j}(\underline{x}^{N-j}, N-j)$ was defined to be the minimum of the sum of the j interval contributions to the criterion. That is

$$\hat{Q}_{N-j}(\underline{x}^{N-j}, N-j) = \min_{\substack{\underline{u}^{N-j} \in U^{N-j} \\ \cdots\cdots \\ \underline{u}^{N-1} \in U^{N-1}}} M\{ \sum_{i=1}^{j} G^{N-i}(\underline{x}^{N-i}, \underline{u}^{N-i}, N-i)$$

$$|\underline{x}^{N-i-1}, \underline{u}^{N-i-1}\}$$

where the conditioning is only applicable for states following $\underline{x}^{N-j}$.

To incorporate reliability effects into the above, the definition of $\hat{Q}$ is enlarged to incorporate a probability of failure parameter in the argument of $\hat{Q}$, additional to the state vector and discretization point arguments. This notation explicitly shows the dependence on the value of reliability at each interval. In particular $\hat{Q}_{N-j}(\underline{x}^{N-j}, \rho^{N-j}, N-j)$ is defined as the minimum of the sum of the last j interval contributions to the criterion which gives an overall probability of failure of ρ^{N-j}, $(0 \leq \rho^{N-j} \leq P_F)$, at the (N-j)'th discretization point to which the state $\underline{x}^{N-j}$ refers. Equation (n.3.4) clearly becomes

$$\hat{Q}_{N-j}(\underline{x}^{N-j}, \rho^{N-j}, N-j) = \min_{\substack{\underline{u}^{N-j} \in U^{N-j} \\ \eta^{N-j} \leq \rho^{N-j}}} \{\int G^{N-j}(\underline{x}^{N-j}, \underline{u}^{N-j}, N-j)$$

$$p(\underline{x}^{N-j})d\underline{x}^{N-j} \qquad \text{(n.3.5)}$$

$$+ \int \hat{Q}_{N-j+1}(\underline{x}^{N-j+1}, \rho^{N-j+1}, N-j+1)p(\underline{x}^{N-j+1}|\underline{x}^{N-j}, \underline{u}^{N-j})d\underline{x}^{N-j+1}\}$$

This is the required optimality condition (recurrence relation) for the solution of the constrained problem. Notice that the solution function $\hat{Q}_{N-j}$ is defined over an extra dimension to $\hat{Q}_{N-j}$ of the 'unconstrained' problem (§M).

The solution routine proceeds along similar lines to article §M.3. Starting with the last interval between y = (N-1)Δ and y = NΔ,

$$\hat{Q}_{N-1}(\underline{x}^{N-1}, \rho^{N-1}, N-1) = \min_{\substack{\underline{u}^{N-1} \in U^{N-1} \\ \eta^{N-1} \leq \rho^{N-1}}} \{M[G^{N-1}(\underline{x}^{N-1}, \underline{u}^{N-1}, N-1)]\} \qquad \text{(n.3.6)}$$

Equation (n.3.6) is thus solely a function of $\underline{x}^{N-1}$, ρ^{N-1} and $\underline{u}^{N-1}$ from which $\hat{\underline{u}}^{N-1}$ may be obtained as a function of the other two by a minimization procedure. The pair $(\hat{Q}_{N-1}, \hat{\underline{u}}^{N-1})$ is thus determined as in the preceding derivation.

For the last two intervals between $y = (N-2)\Delta$ and $y = N\Delta$,

$$\hat{Q}_{N-2}(\underline{x}^{N-2}, \rho^{N-2}, N-2) = \min_{\substack{\underline{u}^{N-2} \in U^{N-2} \\ \eta^{N-2} \leq \rho^{N-2}}} \{M[G^{N-2}] + \int \hat{Q}_{N-1}(\underline{x}^{N-1}, \rho^{N-1}, N-1)\, p(\underline{x}^{N-1}|\underline{x}^{N-2}, \underline{u}^{N-2})\, d\underline{x}^{N-1}\} \tag{n.3.7}$$

where $\underline{x}^{N-1} = \underline{x}^{N-2} + \underline{f}[\underline{x}^{N-2}, \underline{u}^{N-2}, N-2]$

$\rho^{N-1} = \rho^{N-2} - \eta^{N-2}$ $\qquad 0 \leq \rho^{N-1} \leq P_F$

from which the pair $(\hat{Q}_{N-2}, \hat{\underline{u}}^{N-2})$ may be obtained.

The procedure is repeated for the remaining intervals and differs from the previous solution procedure only in the additional dimension contributed by the constraint.

Formal numerical computations would possibly proceed in the following manner (see for example Bellman (1961) for the like deterministic form):

(i) For the last interval, using a series of values of admissible $\underline{u}$, the minimization of (n.3.6) may be carried out for given characteristics (for example moments or the probability law) of $\underline{x}^{N-1}$. The range of characteristics of $\underline{x}^{N-1}$ (which necessarily defines an associated η^{N-1}) that may be chosen are such that $0 \leq \eta^{N-1} \leq \rho^{N-1}$ where $0 \leq \rho^{N-1} \leq P_F$ and where to sufficient accuracy, for small probabilities, $\rho = \sum_i \eta^i$. This reduces to $\rho^{N-1} = \eta^{N-1}$ for one summand (discretization point).) $\hat{Q}_{N-1}$ may then be stored as a function of the local characteristics of state $\underline{x}^{N-1}$ and the cumulative reliability ρ^{N-1}.

(ii) Passing to the second last interval, this knowledge of $\hat{Q}_{N-1}$ for various characteristics of $\underline{x}^{N-1}$, and ρ^{N-1} may be used to obtain a similar link between $\hat{Q}_{N-2}$ and the characteristics of $\underline{x}^{N-2}$, and ρ^{N-2}. Again taking a series of values of $\underline{u}$ within the permitted range, the minimization of (n.3.7) may be completed for a given characterization of $\underline{x}^{N-2}$. An ordinary search procedure may be suitable in this case to find the minimum control; this process is repeated for further characterizations of $\underline{x}^{N-2}$ over the relevant range of this vector. The choice of the characterizations of $\underline{x}^{N-2}$ is restricted to values which give $\eta^{N-1} + \eta^{N-2} \leq P_F$. Notice that a particular characterization of $\underline{x}^{N-2}$ and a series of values of $\underline{u}^{N-2}$ are all that are needed to evaluate the first term on the right hand side of (n.3.7), namely G^{N-2}. However for the second term on the right hand side of (n.3.7), the system equations, $\underline{x}^{N-1} = \underline{x}^{N-2} + \underline{f}[\underline{x}^{N-2}, \underline{u}^{N-2}, N-2]$, and the reliability equation, either $\sum_{i=N-1}^{N-2} \eta^{i} = \rho^{N-2}$ or $\rho^{N-1} = \rho^{N-2} - \eta^{N-2}$, were needed to relate the $\hat{Q}_{N-1}$: characterization of $\underline{x}^{N-1}:\rho^{N-1}$ values of the last interval to the present interval. $\hat{Q}_{N-2}$ may now overwrite $\hat{Q}_{N-1}$ in storage as $\hat{Q}_{N-1}$ is no longer required.

(iii) The computations involved in the transition from N-1 to N-2 described in (ii) are sufficiently general and may be used in going from the N-j to the N-j-1 stages ending in stored values for $\hat{Q}_0$ for a set of characterizations of $\underline{x}^0$, and ρ^0. Using the end-state conditions and the definition of $\rho^0 = P_F$, the calculations may be reworked in the reverse direction (that is positive y direction) to find the optimal controls $\hat{\underline{u}}^0, \ldots, \hat{\underline{u}}^{N-j}, \ldots, \hat{\underline{u}}^N$.

N.4 COMMENT.

Reliability constraints, when understood to be associated with nonchanging (nonlearning, nonadaptive) systems, may be readily incorporated into the stochastic optimality conditions derived in section §M. Through this, a useful solution procedure has been derived for optimal structures incorporating a probabilistic measure of safety.

The derivation is independent of the statistical characteristics of the variables concerned and hence may be assumed to have wide applicability. The extension to multiple reliability constraints, multiple loadings and others is apparent and conceptually involves no new ideas although the solution procedure is then definitely far more complex computationally. The deficiencies of the approach remain the same as outlined in section §M with the additional feature of working in an extra dimension.

It appears that the approach would only be feasible for elementary problems and it is doubtful if it would have much use at the level of involvedness of conventional design problems in its present form. The problem of the design of continuous structures with reliability constraints plainly appears too difficult, but the basis of the solution has been outlined here for future reference. Where a design problem is insoluble it may in certain cases, for comparison purposes, be worthwhile deriving the solution for a simpler problem in order to develop some 'feel' for the original problem.

§O AN ILLUSTRATION IN STOCHASTIC DESIGN

0.1 GENERAL DESCRIPTION.

To clarify the various stages involved in the derivation of the conditions for optimality (stochastic) consider the elementary beam equation with conventional Bernoulli-Euler assumptions:

$$\frac{d^2}{dy^2}\left[EI(y)\,\frac{d^2w(y)}{dy^2}\right] = q(y) \tag{o.1.1}$$

It is assumed that the loading is random, and at any position y, its first and second moments (probabilistic) are known.

An integral square error criterion has been chosen for which the optimum flexibility (rigidity) of the beam is sought. The criterion is a measure of the deviations of the state and control from certain desired values, which may be chosen according to the problem at hand. Essentially the problem is thus one of selecting a control such that the system follows a desired behaviour and assumes a certain geometry as closely as possible.

Introducing the state vector $\underline{x} = (x_1, x_2, x_3, x_4)^T$, and control u;

$$\begin{bmatrix} x_1 \triangleq w \\[2mm] x_2 \triangleq \dfrac{dw}{dy} \\[2mm] x_3 \triangleq EI\,\dfrac{d^2w}{dy^2} \\[2mm] x_4 \triangleq \dfrac{d}{dy}\left(EI\,\dfrac{d^2w}{dy^2}\right) \end{bmatrix}$$

$$u \triangleq 1/EI$$

Upon differentiating with respect to y, the system equation (o.1.1) may be rewritten in the form;

$$\text{(o.1.2)} \qquad \left[\begin{array}{l} \dfrac{dx_1}{dy} = x_2 \\[2em] \dfrac{dx_2}{dy} = ux_3 \\[2em] \dfrac{dx_3}{dy} = x_4 \\[2em] \dfrac{dx_4}{dy} = q \end{array}\right]$$

The fourth order equation has consequently been reduced to four first order equations by a suitable choice of the new state variables x_i, $i = 1, 2, 3, 4$.

The integral square error measure has been chosen in the general form

$$\text{(o.1.3)} \qquad \tilde{Q} = \int_0^L \left\{\alpha(y)\,[x_1(y) - x_1^d(y)]^2 + \beta(y)\,[u(y) - u^d(y)]^2\right\}dy$$

where $x_1^d(y)$ and $u^d(y)$ denote the desired state (deflection) and flexibility of the beam, corresponding to the state $x_1(y)$ and control $u(y)$ respectively, and in general will be deterministic functions. The measure penalises large deviations from the desired values more heavily than small deviations. $\alpha(y)$ and $\beta(y)$ are weighting factors which indicate the relative importance of the various terms in the error measure. In any particular design, these weighting factors may be selected to satisfy specified performance requirements, physical constraints or other design briefs. A particular case of interest is when $\alpha = 0$ and u^d large; the criterion is then equivalent to minimizing the stiffness (which is related to the weight and material) of the beam.

For random arguments of the integrand, the appropriate deterministic measure of system suitability is the expected value of $\tilde{Q}$, $M\{\tilde{Q}\}$.

O.2 DISCRETIZATION SCHEME.

The length of the beam, the interval $[0,L]$, is divided into N equal subintervals $\Delta = L/N$, such that at any position $y = k\Delta$, $k = 0, 1, \ldots, N$, the system equations (o.1.2) may be restated in the discrete form

$$\text{(o.2.1)} \qquad \begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \\ x_3^{k+1} \\ x_4^{k+1} \end{bmatrix} = \begin{bmatrix} x_1^{k} \\ x_2^{k} \\ x_3^{k} \\ x_4^{k} \end{bmatrix} + \Delta \begin{bmatrix} x_2^{k} \\ x_3^{k} u^{k} \\ x_4^{k} \\ q^{k} \end{bmatrix} \qquad k = 0, 1, \ldots, N-1$$

according to the discretization procedure outlined in article §E.2.

The known expected values and variances of the random variables q^k, $k = 0, 1, \ldots, N$ will be denoted by $M\{q^k\} = \theta^k$ and $D\{q^k\} = (\sigma^k)^2$ respectively and are assumed independent for each k.

Boundary conditions, chosen as deterministic for the illustration (although this is not essential to the approach), may be expressed as Dirac delta functions of the state. In particular

$$\text{(o.2.2)} \qquad \begin{aligned} p(x_1^0) &= \delta(x_1^0-0) \\ p(x_2^0) &= \delta(x_2^0-0) \\ p(x_3^N) &= \delta(x_3^N-0) \\ p(x_4^N) &= \delta(x_4^N-Z) \end{aligned}$$

where Z is some deterministic end reaction.

The criterion may also be expressed in discrete form as a finite summation;

$$\text{(o.2.3)} \qquad Q = M\{\tilde{Q}\} = M\{\Delta \sum_{k=0}^{N-1} \alpha^k [x_1^k - x_1^{d(k)}]^2 + \beta^k [u^k - u^{d(k)}]^2\}$$

O.3 DESIGN PROBLEM STATEMENT.

The design problem may now be stated as follows: To determine the set of controls $\{u^k;\ k = 0, 1, \ldots, N-1\}$ for the system behaving according to the equations (o.2.1), with boundary conditions (o.2.2), such that the expected value of $\tilde{Q}$, given by expression (o.2.3) is minimized.

O.4 COMPUTATION OF THE OPTIMAL DESIGN.

For the last subinterval extending between $y = (N-1)\Delta$ and $y = N\Delta$

$$\hat{Q}_{N-1}(\underline{x}^{N-1}, N-1) = \min_{u^{N-1}} M\{\Delta\alpha^{N-1}[x_1^{N-1} - x_1^{d(N-1)}]^2 + \Delta\beta^{N-1}[u^{N-1} - u^{d(N-1)}]^2\}$$

Using the variance relationship

(a) $$D\{X\} = M\{X^2\} - [M\{X\}]^2$$

and the linear property of the expectation operation, namely

(b) $$M\{aX + bY\} = aM\{X\} + bM\{Y\}$$

then,

$$\hat{Q}_{N-1}(\underline{x}^{N-1}, N-1) = \min_{u^{N-1}} \{\Delta\alpha^{N-1}[x_1^{N-1} - x_1^{d(N-1)}]^2 + \Delta\beta^{N-1}[u^{N-1} - u^{d(N-1)}]^2\}$$

Notice that even though the state is random there is no variance term in x_1 in this line as at this stage x_1^{N-1} is assumed known for the purpose of obtaining u^{N-1}.

Clearly $\hat{u}^{N-1} = u^{d(N-1)}$ and $\hat{Q}_{N-1}(\underline{x}^{N-1}, N-1) = \Delta\alpha^{N-1}[x_1^{N-1} - x_1^{d(N-1)}]^2$.

For the last two subintervals between $y = (N-2)\Delta$ and $y = N\Delta$

$$\hat{Q}_{N-2}(\underline{x}^{N-2}, N-2) = \min_{u^{N-2}} \{ M\{\Delta\alpha^{N-2}[x_1^{N-2} - x_1^{d(N-2)}]^2 + \Delta\beta^{N-2}[u^{N-2} - u^{d(N-2)}]^2\}$$
$$+ M\{\Delta\alpha^{N-1}[x_1^{N-1} - x_1^{d(N-1)}]^2 | \underline{x}^{N-2}, u^{N-2}\}\}$$

$$= \min_{u^{N-2}} \{\Delta\alpha^{N-2}[x_1^{N-2} - x_1^{d(N-2)}]^2 + \Delta\beta^{N-2}[u^{N-2} - u^{d(N-2)}]^2$$

$$+ \Delta\alpha^{N-1}[x_1^{N-2} + \Delta x_2^{N-2} - x_1^{d(N-1)}]^2\}$$

Clearly $\hat{u}^{N-2} = u^{d(N-2)}$ and

$$\hat{Q}_{N-2}(\underline{x}^{N-2}, N-2) = \Delta\alpha^{N-2}[x_1^{N-2} - x_1^{d(N-2)}]^2 + \Delta\alpha^{N-1}[x_1^{N-2} + \Delta x_2^{N-2} - x_1^{d(N-1)}]^2$$

It follows that for the last and second last subintervals, the optimal control derived, implies choosing a 'flexibility' equal to the desired 'flexibility' and independent of the value of the state over these regions.

For the last three subintervals

$$\hat{Q}_{N-3}(\underline{x}^{N-3}, N-3) = \min_{u^{N-3}} \{ M\{\Delta\alpha^{N-3}[x_1^{N-3} - x_1^{d(N-3)}]^2 + \Delta\beta^{N-3}[u^{N-3} - u^{d(N-3)}]^2\}$$
$$+ M\{\Delta\alpha^{N-2}[x_1^{N-2} - x_1^{d(N-2)}]^2 + \Delta\alpha^{N-1}[x_1^{N-2} + \Delta x_2^{N-2} - x_1^{d(N-1)}]^2$$
$$|\underline{x}^{N-3}, u^{N-3}\}\}$$

$$= \min_{u^{N-3}} \{\Delta\alpha^{N-3}[x_1^{N-3} - x_1^{d(N-3)}]^2 + \Delta\beta^{N-3}[u^{N-3} - u^{d(N-3)}]^2$$
$$+ \Delta\alpha^{N-2}[x_1^{N-3} + \Delta x_2^{N-3} - x_1^{d(N-2)}]^2$$
$$+ \Delta\alpha^{N-1}[x_1^{N-3} + 2\Delta x_2^{N-3} + \Delta^2 u^{N-3}(\kappa^{N-3}\theta^{N-3} + \gamma^{N-3}) - x_1^{d(N-1)}]^2$$
$$+ \Delta^5\alpha^{N-1}(\kappa^{N-3}\sigma^{N-3})^2(u^{N-3})^2\}$$

where the expressions (a) and (b), above, have been used in the last line. κ^{N-3} and γ^{N-3} are constants.

The expression within the braces is stationary with respect to u^{N-3} when

$$0 = 2\Delta\beta^{N-3}[\hat{u}^{N-3} - u^{d(N-3)}] + 2\Delta\alpha^{N-1}[x_1^{N-3} + 2\Delta x_2^{N-3}$$
$$+ \Delta^2\hat{u}^{N-3}(\kappa^{N-3}\theta^{N-3} + \gamma^{N-3}) - x_1^{d(N-1)}]\ \Delta^2(\kappa^{N-3}\theta^{N-3} + \gamma^{N-3})$$

$$+ 2\Delta^5\alpha^{N-1}(\kappa^{N-3}\sigma^{N-3})^2\hat{u}^{N-3}$$

or $$\hat{u}^{N-3} = \frac{\beta^{N-3}u^{d(N-3)} - \Delta^2(\kappa^{N-3}\theta^{N-3} + \gamma^{N-3})\alpha^{N-1}[x_1^{N-3} + 2\Delta x_2^{N-3} - x_1^{d(N-1)}]}{\beta^{N-3} + \Delta^4\alpha^{N-1}(\kappa^{N-3}\theta^{N-3} + \gamma^{N-3})^2 + \Delta^4\alpha^{N-1}(\kappa^{N-3}\sigma^{N-3})^2}$$

which is of the form $\hat{u}^{N-3} = C^{N-3} + D_1^{N-3}x_1^{N-3} + D_2^{N-3}x_2^{N-3}$

where C^{N-3}, D_1^{N-3} and D_2^{N-3} are constants given by

$$C^{N-3} = \frac{\beta^{N-3}u^{d(N-3)} + \Delta^2(\kappa^{N-3}\theta^{N-3} + \gamma^{N-3})\alpha^{N-1}x_1^{d(N-1)}}{\beta^{N-3} + \Delta^4\alpha^{N-1}(\kappa^{N-3}\theta^{N-3} + \gamma^{N-3})^2 + \Delta^4\alpha^{N-1}(\kappa^{N-3}\sigma^{N-3})^2}$$

$$D_1^{N-3} = \frac{-\Delta^2(\kappa^{N-3}\theta^{N-3} + \gamma^{N-3})\alpha^{N-1}}{\beta^{N-3} + \Delta^4\alpha^{N-1}(\kappa^{N-3}\theta^{N-3} + \gamma^{N-3})^2 + \Delta^4\alpha^{N-1}(\kappa^{N-3}\sigma^{N-3})^2}$$

$$D_2^{N-3} = \frac{-2\Delta^3(\kappa^{N-3}\theta^{N-3} + \gamma^{N-3})\alpha^{N-1}}{\beta^{N-3} + \Delta^4\alpha^{N-1}(\kappa^{N-3}\theta^{N-3} + \gamma^{N-3})^2 + \Delta^4\alpha^{N-1}(\kappa^{N-3}\sigma^{N-3})^2}$$

and

$$\begin{aligned}\hat{Q}_{N-3}(\underline{x}^{N-3}, N-3) = {} & \Delta\alpha^{N-3}[x_1^{N-3} - x_1^{d(N-3)}]^2 + \Delta\beta^{N-3}[C^{N-3} + D_1^{N-3}x_1^{N-3} \\ & + D_2^{N-3}x_2^{N-3} - u^{d(N-3)}]^2 + \Delta\alpha^{N-2}[x_1^{N-3} + \Delta x_2^{N-3} - x_1^{d(N-2)}]^2 \\ & + \Delta\alpha^{N-1}[x_1^{N-3} + 2\Delta x_2^{N-3} + \Delta^2(\kappa^{N-3}\theta^{N-3} + \gamma^{N-3})(C^{N-3} + D_1^{N-3}x_1^{N-3} \\ & + D_2^{N-3}x_2^{N-3}) - x_1^{d(N-1)}]^2 \\ & + \Delta^5\alpha^{N-1}(\kappa^{N-3}\sigma^{N-3})^2(C^{N-3} + D_1^{N-3}x_1^{N-3} + D_2^{N-3}x_2^{N-3})^2\end{aligned}$$

which is of the general form

$$\begin{aligned}\hat{Q}_{N-3}(\underline{x}^{N-3}, N-3) = {} & R_0^{N-3} + R_1^{N-3}x_1^{N-3} + R_2^{N-3}x_2^{N-3} + S_{11}^{N-3}(x_1^{N-3})^2 + S_{12}^{N-3}(x_1^{N-3}x_2^{N-3}) \\ & + S_{22}^{N-3}(x_2^{N-3})^2\end{aligned}$$

where R_0^{N-3}, R_1^{N-3}, R_2^{N-3}, S_{11}^{N-3}, S_{12}^{N-3} and S_{22}^{N-3} are constants. In fact it can be shown that $\hat{Q}_{N-j}$ is of a similar form for any j. So that in general

$$\hat{Q}_{N-j}(\underline{x}^{N-j}, N-j) = R_0^{N-j} + R_1^{N-j}x_1^{N-j} + R_2^{N-j}x_2^{N-j} + S_{11}^{N-j}(x_1^{N-j})^2 + S_{12}^{N-j}(x_1^{N-j}x_2^{N-j}) + S_{22}^{N-j}(x_2^{N-j})^2$$

$$= \min_{u^{N-j}} \{ M\{\Delta\alpha^{N-j}[x_1^{N-j} - x_1^{d(N-j)}]^2 + \Delta\beta^{N-j}[u^{N-j} - u^{d(N-j)}]^2\} + M\{R_0^{N-j+1} + R_1^{N-j+1}x_1^{N-j+1} + R_2^{N-j+1}x_2^{N-j+1} + S_{11}^{N-j+1}(x_1^{N-j+1})^2 + S_{12}^{N-j+1}(x_1^{N-j+1}x_2^{N-j+1}) + S_{22}^{N-j+1}(x_2^{N-j+1})^2 | \underline{x}^{N-j}, u^{N-j}\}\}$$

(o.4.1)

$$= \min_{u^{N-j}} \{\Delta\alpha^{N-j}[x_1^{N-j} - x_1^{d(N-j)}]^2 + \Delta\beta^{N-j}[u^{N-j} - u^{d(N-j)}]^2 + R_0^{N-j+1} + R_1^{N-j+1}(x_1^{N-j} + \Delta x_2^{N-j}) + R_2^{N-j+1}(x_2^{N-j} + \Delta(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j})u^{N-j}) + S_{11}^{N-j+1}(x_1^{N-j} + \Delta x_2^{N-j})^2 + S_{12}^{N-j+1}(x_1^{N-j} + \Delta x_2^{N-j})(x_2^{N-j} + \Delta(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j})u^{N-j}) + S_{22}^{N-j+1}(x_2^{N-j} + \Delta(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j})u^{N-j})^2 + S_{22}^{N-j+1}(\Delta\kappa^{N-j}\sigma^{N-j}u^{N-j})^2\}$$

Differentiating the expression within the braces with respect to u^{N-j} and then setting to zero for the stationary value

$$0 = 2\Delta\beta^{N-j}[\hat{u}^{N-j} - u^{d(N-j)}] + R_2^{N-j+1}\Delta(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j}) + S_{12}^{N-j+1}(x_1^{N-j} + \Delta x_2^{N-j})\ \Delta(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j}) + 2S_{22}^{N-j+1}(x_2^{N-j} + \Delta(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j})\hat{u}^{N-j})\ \Delta(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j}) + 2S_{22}^{N-j+1}(\Delta\kappa^{N-j}\sigma^{N-j})^2\hat{u}^{N-j}$$

and $\hat{u}^{N-j}$ is of the form

(o.4.2a) $$\hat{u}^{N-j} = C^{N-j} + D_1^{N-j}x_1^{N-j} + D_2^{N-j}x_2^{N-j}$$

where

$$C^{N-j} = \frac{2\Delta\beta^{N-j}\, u^{d(N-j)} - R_2^{N-j+1}\,\Delta(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j})}{2\Delta\beta^{N-j} + 2S_{22}^{N-j+1}\,\Delta^2(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j})^2 + 2S_{22}^{N-j+1}(\Delta\kappa^{N-j}\sigma^{N-j})^2}$$

(o.4.2b) $$D_2^{N-j} = \frac{-S_{12}^{N-j+1}\,\Delta(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j})}{2\Delta\beta^{N-j} + 2S_{22}^{N-j+1}\,\Delta^2(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j})^2 + 2S_{22}^{N-j+1}(\Delta\kappa^{N-j}\sigma^{N-j})^2}$$

$$D_2^{N-j} = \frac{-S_{12}^{N-j+1}\,\Delta^2(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j}) - 2S_{22}^{N-j+1}\,\Delta(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j})}{2\Delta\beta^{N-j} + 2S_{22}^{N-j+1}\,\Delta^2(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j})^2 + 2S_{22}^{N-j+1}(\Delta\kappa^{N-j}\sigma^{N-j})^2}$$

Substituting (o.4.2a) back into (o.4.1)

$$R_0^{N-j} + R_1^{N-j}x_1^{N-j} + R_2^{N-j}x_2^{N-j} + S_{11}^{N-j}(x_1^{N-j})^2 + S_{12}^{N-j}(x_1^{N-j}x_2^{N-j}) + S_{22}^{N-j}(x_2^{N-j})^2$$

$$= \Delta\alpha^{N-j}[x_1^{N-j} - x_1^{d(N-j)}]^2 + \Delta\beta^{N-j}[C^{N-j} + D_1^{N-j}x_1^{N-j} + D_2^{N-j}x_2^{N-j}$$

$$- u^{d(N-j)}]^2 + R_0^{N-j+1} + R_1^{N-j+1}(x_1^{N-j} + \Delta x_2^{N-j})$$

$$+ R_2^{N-j+1}(x_2^{N-j} + \Delta(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j})[C^{N-j} + D_1^{N-j}x_1^{N-j} + D_2^{N-j}x_2^{N-j}])$$

(o.4.3) $$+ S_{11}^{N-j+1}(x_1^{N-j} + \Delta x_2^{N-j})^2$$

$$+ S_{12}^{N-j+1}(x_1^{N-j} + \Delta x_2^{N-j})(x_2^{N-j} + \Delta(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j})[C^{N-j} + D_1^{N-j}x_1^{N-j}$$

$$+ D_2^{N-j}x_2^{N-j}]) + S_{22}^{N-j+1}(x_2^{N-j} + \Delta(\kappa^{N-j}\theta^{N-j} + \gamma^{N-j})[C^{N-j} + D_1^{N-j}x_1^{N-j}$$

$$+ D_2^{N-j}x_2^{N-j}])^2 + S_{22}^{N-j+1}(\Delta\kappa^{N-j}\sigma^{N-j}[C^{N-j} + D_1^{N-j}x_1^{N-j} + D_2^{N-j}x_2^{N-j}])^2$$

Equations (o.4.2) and (o.4.3) define an iterative process working backwards from the last interval. The computations may be started with the previously derived values for j=1, namely

$$R_0^{N-1} = \Delta\alpha^{N-1}(x_1^{d(N-1)})^2 \qquad C^{N-1} = u^{d(N-1)}$$

$$R_1^{N-1} = -\,2\Delta\alpha^{N-1}x_1^{d(N-1)} \qquad D_1^{N-1} = 0$$

(o.4.4) $$R_2^{N-1} = 0 \qquad D_2^{N-1} = 0$$

$$S_{11}^{N-1} = \Delta\alpha^{N-1}$$

$$S_{12}^{N-1} = 0$$

$$S_{22}^{N-1} = 0$$

These are used to calculate in turn

(i) C^{N-2}, D_1^{N-2} and D_2^{N-2} from equation (o.4.2b)

(ii) R_0^{N-2}, R_1^{N-2}, R_2^{N-2}, S_{11}^{N-2}, S_{12}^{N-2} and S_{22}^{N-2} from equation (o.4.3) by matching coefficients of like states on the left and right hand sides,

and iterating to N-3 and so on up to j = N. Equations (o.4.2) and (o.4.3) with initial conditions (o.4.4) represent a 'closed form' solution to the problem.

After calculating the pairs $(\hat{u}^{N-1}, \hat{Q}_{N-1})$, ... , $(\hat{u}^{N-j}, \hat{Q}_{N-j})$, ... , $(\hat{u}^{0}, \hat{Q}_{0})$, as functions of the state variables, the boundary conditions are introduced and the successive states $\underline{x}^0$, ... , $\underline{x}^{N-j}$, ... , $\underline{x}^N$ may be computed iteratively from the system equations as the optimal controls become known at each stage.

As an illustration of the iterative computations, the results are summarized in tables 0.4.1 and 0.4.2 for the following numerical values:

loading: θ = 0.150 kip/ft

σ^2 = (0.035 kip/ft)^2 = 0.001225 (kip/ft)^2

end reaction: Z = 0.550 kip

length: L = 10 ft

weighting factors: $\beta = 2\alpha$

desired deflection: x_1^d = 0.003500 ft

desired 'flexibility': u^d = 0.000450 (kip ft^2)$^{-1}$

intervals: 10

From table 0.4.1 it is evident that the total 'cost' of the design, $\hat{Q}_{N-10}$ = 0.000008. Notice that there is no entry for u^N as u applies between discretization points and changes in a step manner at these points.

j	C^{N-j}	D_1^{N-j}	D_2^{N-j}	R_0^{N-j}	R_1^{N-j}	R_2^{N-j}	S_{11}^{N-j}	S_{12}^{N-j}	S_{22}^{N-j}
10	0.000266	-0.073204	-0.354458	0.000008	-0.029689	-0.055853	4.320847	16.028703	30.740623
9	0.000368	-0.097726	-0.435836	0.000037	-0.034088	-0.076959	4.891550	21.456553	41.218383
8	0.000374	-0.091250	-0.357416	0.000059	-0.036221	-0.081466	5.104645	22.067154	38.173564
7	-0.000212	0.076603	0.260754	0.000056	-0.031544	-0.059025	4.387676	15.509073	22.619231
6	-0.000948	0.271711	0.833923	0.000045	-0.024879	-0.035284	3.448166	9.024569	10.847389
5	-0.001224	0.343549	0.987252	0.000040	-0.021423	-0.024741	2.955969	6.229685	6.445076
4	-0.001322	0.392597	1.011508	0.000038	-0.020109	-0.020970	2.753578	5.306675	4.971517
3	-0.000844	0.327684	0.655367	0.000035	-0.019354	-0.017706	2.680509	4.722034	3.722033
2	0.000450			0.000024	-0.014000	-0.007000	2.000000	2.000000	1.000000
1	0.000450			0.000012	-0.007000		1.000000		
0									

Table 0.4.1 Illustration Constants.

j	$\hat{u}^{N-j}$ $(\text{kip ft}^2)^{-1}$	$\hat{x}_1^{N-j}$ $\frac{\text{ft}}{\text{ft}^2}$	$\hat{x}_2^{N-j}$ $\frac{\text{ft/ft}}{(\text{ft/ft})^2}$	$\hat{x}_3^{N-j}$ $\frac{\text{kip ft}}{(\text{kip ft})^2}$	$\hat{x}_4^{N-j}$ $\frac{\text{kip ft/ft}}{(\text{kip ft/ft})^2}$
10	0.000266			2.000000 3.062500	-0.950000 0.122500
9	0.000136		0.000532 0.002167×10^{-4}	1.125000 2.009306	-0.800000 0.099225
8	0.000081	0.000532 0.002167×10^{-4}	0.000685 0.004333×10^{-4}	0.400000 1.254400	-0.650000 0.078400
7	0.000068	0.001217 0.012629×10^{-4}	0.000717 0.005610×10^{-4}	-0.175000 0.735306	-0.500000 0.060025
6	0.000165	0.001934 0.035073×10^{-4}	0.000705 0.006517×10^{-4}	-0.600000 0.396900	-0.350000 0.044100
5	0.000281	0.002639 0.071829×10^{-4}	0.000606 0.008304×10^{-4}	-0.875000 0.191406	-0.200000 0.030625
4	0.000316	0.003245 0.128978×10^{-4}	0.000360 0.010696×10^{-4}	-1.000000 0.078400	-0.050000 0.019600
3	0.000366	0.003605 0.213957×10^{-4}	0.000044 0.012604×10^{-4}	-0.975000 0.024806	0.100000 0.011025
2	0.000450	0.003649 0.330421×10^{-4}	-0.000313 0.013932×10^{-4}	-0.800000 0.004900	0.250000 0.004900
1	0.000450	0.003336 0.480048×10^{-4}	-0.000673 0.014685×10^{-4}	-0.475000 0.000306	0.400000 0.001225
0		0.002663 0.662656×10^{-4}	-0.000887 0.014877×10^{-4}		0.550000

Table 0.4.2 Illustration Solution (upper value = expected value, lower value = variance).

$\underset{\sim}{x}$ applies at discretization points and hence has an entry on each line in table 0.4.2.

0.5 DISCUSSION.

Where a solution in a 'closed form', as represented by equations (o.4.2) and o.4.3), is not sought or is difficult to find, a method of calculation which is suitable for general digital computer application would be desirable. A possible method (see for example Bellman (1961) for the like deterministic form) would proceed as follows:

(i) $\hat{Q}_{N-1}$, it will be recalled, was found to be a function of the local state $\underset{\sim}{x}^{N-1}$ and may be stored for a number of discrete values of $\underset{\sim}{x}^{N-1}$ over the range of interest of this vector. Interpolation or extrapolation procedures may be employed to find other values not stored.

(ii) Passing to the next subinterval, $\hat{Q}_{N-2}$ is found as

$$\text{(o.5.1)} \qquad \hat{Q}_{N-2} = \min_{u^{N-2}} \{ M[G^{N-2}(\underset{\sim}{x}^{N-2}, u^{N-2})] + M[\hat{Q}_{N-1} | \underset{\sim}{x}^{N-2}, u^{N-2}] \}$$

Since $\hat{Q}_{N-1}$ is known for any numerical value of $\underset{\sim}{x}^{N-1}$, the minimization of (o.5.1) may be carried out over the permissible range of u^{N-2} for any given $\underset{\sim}{x}^{N-2}$. An ordinary search procedure may be suitable in this case to find the minimum, the process of which is repeated for many values of $\underset{\sim}{x}^{N-2}$ over the relevant range of this vector. $\hat{Q}_{N-2}$ may now overwrite $\hat{Q}_{N-1}$ in storage as $\hat{Q}_{N-1}$ is no longer required.

(iii) The computations involved in the transition from N-1 to N-2 just described are sufficiently general and may be used in going from the N-j to the N-j-1 stages ending in stored values for $\hat{Q}_0$ for a set of values of $\underset{\sim}{x}^0$. Using the end-state conditions, the calculations may now be reworked in the reverse direction to find $\hat{u}^0, \ldots, \hat{u}^{N-j}, \ldots, \hat{u}^N$.

With such a store and search procedure, the computations may be simplified if direct constraints on the state and/or control are present. The above procedure assumes that the expectations of the relevant functions can be evaluated explicitly; this should always be possible, if only approximately, in any problem. Where the expectations are not evaluated explicitly, the above procedure is modified at each stage by replacing the state with the characteristics (for example moments, probability law) of the state.

It will be apparent that there could be quite demanding requirements of storage and time involved in the computations. However in spite of this, the approach does provide a reasonably elegant path for the design of optimum structures with stochastic influences. Certain limitations may also be apparent in the size of the problem that may be treated. It appears that at the present stage, large designs are not feasible and still remain a major problem. Stochastic design problems represent an order of magnitude greater difficulty over their deterministic counterparts. The practicability of the design procedure above lies in exploiting the iterative equations on a digital computer. For any particular problem the equations may be readily programmed.

As a result of discretization, the solutions only serve as an approximation to the solutions of the original continuous problem. The discretization scheme however is based on central difference approximations which are expected to improve as the interval size tends to zero.

The approach solves a family of problems in obtaining the optimal solution; results are found for all possible initial boundary states. This in effect is equivalent to a sensitivity analysis of the problem, where the effect of changes in the boundary states may be evaluated. The optimal solution is 'imbedded' in the range of solutions for all possible states.

PART 4

CLOSURE, REFERENCES AND APPENDICES

§P CLOSURE

General: The ordered conceptual framework basic to control systems theory was shown to offer a unifying approach to structural modelling and design for the range of applications considered. To produce the desired level of generality, a certain amount of abstraction in the mathematical sense was required. In particular a state space formalism offered the foundation for a study in single-level modelling with a definite extension to multilevel modelling and the prospect of elucidating the hierarchical composition of structures. Control theory, using the idea of state, has developed a theory useful as regards to descriptive qualities and functional understanding of systems and also in the solution of complex problems. This approach to structural systems is very new, there being very few results reported in the literature.

The treatment related to a particular subset of the total set of modelling and design problems for structures. However the construction of a general systems theory for all structures appears feasible in the very near future. The modelling portion of the theory would almost certainly employ hierarchical multilevel ideas and control-state notions, both aspects of which have been shown to be clearly fundamental to structures. The establishment of a general theory of optimal design of structures (already available in various special forms) is also imminent. Thinking akin to optimal control systems theory philosophy would provide the general solution techniques.

Modelling: The underlying modelling approach followed an essentially inductive course by recognising, clarifying and extending basic attributes exhibited by certain elementary structures to a reasonably broad class of structures. The attempt was to establish precise systems definitions and concepts for the single-level structural model. Illustrations enabled the theoretical presentation to be maintained on a relatively formal level.

The models describing the structure were expressed in terms of differential equations. Using state space concepts, the conventional form of the equations of structural mechanics was transformed into a standard first-order form, the so-called state equations. Methods for achieving this transformation were detailed and questions of nonuniqueness of the choice of state were discussed. The state equations provided the generalisation and unification between structures while their first-order form evidently had advantages in both analytical and numerical application.

Both deterministic and stochastic assumptions were modelled and presented side by side with no judgement being made as to which was the more preferable. The two models should coexist in the structures literature, the choice of usage depending on the application. Mathematically and conceptually, determinism was the more tractable, the state of the theory for stochasticism appearing by comparison rather primitive. For the stochastic case, in order to produce tractable computations, the simplifying assumption of Markov properties for the model state was invoked.

The present single-level approach has been confined to systems whose description is continuous over a certain space-time domain. The extension to discrete descriptions is apparent and was sometimes used as an approximate form of the continuous case. The level of the variable descriptions may also be discretized leading to a very interesting characterization of structures, a characterization untried up to the present time but one which may be a fruitful line of investigation. Discrete sample spaces may also be incorporated.

Design: The formulation and solution of the design problem was presented in contexts as general as possible without losing content. It was shown that superficially different problems may be given a common mathematical (and conceptual) basis. The common mathematical basis further permitted solution procedures analogous for all problems. Mathematical transformations between associated problems extended the domain of application.

The underlying approach to optimization was one of establishing the fundamental approach to design rather than the demonstration of the economic or other benefits to be gained from a design solution. The design criteria were formulated in general terms which included the narrower technical interpretations of structural optimization such as minimum weight, deflection and others. The selection and formulation difficulties of the criteria, especially within the context of multiple design requirements, were acknowledged. In this sense a design problem set up consistent with a systems optimization theory has been expounded as opposed to a more specific and subsumed structural optimization viewpoint; various existing synthetic structural procedures, in spite of their essentially individual character, were given a unified basis.

The approach using an optimization format was also helpful in developing understanding in design. This was achieved by studying the basic philopophy of design and emphasizing the nature of the fundamental components of the design process and problem. In these contexts, the motivation for the use of a control framework in optimum design was self evident.

Previous to this work, no computational experience had appeared in the structures literature for the stochastic case, although confidence in treating deterministic systems was well advanced. The present stochastic work should enable a designer to obtain a better feel for the random behaviour of structures. For given equivalent deterministic and stochastic problems, the latter was an order of magnitude more difficult. For problems of equal difficulty (as with the level of difficulty considered in the above), the solutions have been reduced to a stage of mechanical handling and hence permit ready application. However additional research is clearly required before complicated design problems may be treated satisfactorily in a routine manner. Optimal control theory shows the direction in which this future research should point.

For the deterministic design, three complementary derivations were given for the three main structural system models adopted. Each resulted in a distributed parameter extension of Pontryagin's maximum principle but, as the derivation techniques were different, so were the underlying assumptions. The characterization of the optimal solutions (the maximum principles) arose explicitly from the construction of the design problem. The maximum principles defined the properties of the optimal solutions. The results were extended to include singular formulations.

It is believed that realistic design problems may now be tackled and their solution is feasible provided, of course, some form of numerical computation is employed. The prerequisite level of comprehension of the mathematics may however prevent the immediate acceptance and utilization of the results and perhaps may even alienate the applications oriented engineer.

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APPENDIX ONE: ALTERNATIVE DECOMPOSITIONS OF THE SYSTEM EQUATION

Two algorithms are given for the reduction of a general partial differential equation to the form of system type I (equation C.3.1) and type III (equation C.3.3). See Lurie (1963) or Armand (1972) for a general algorithm for type II (equation C.3.2). At the outset it is emphasized that the resulting sets of equations satisfy the mathematical definitions of state and control but will often lack physical meaning. The reductions are offered, firstly, as equivalent reductions cannot be found in the literature and, secondly, as mechanical means of system equation reduction. They are alternative to the reduction schemes proposed elsewhere in the present work (sections §C, §I and §K). Their use is illustrated in sections §G and §K.

System type I: See Wang and Tung (1964) for a special case of this reduction. Consider a general equation in independent variables $\{y_p;\ p = 1, \dots, 4\}$ and dependent variables $\{v_j;\ j = 1, \dots, s\}$;

(1-A1) $$F[y_1, \dots, y_4, \dots, v_j, \dots, \partial_{\underline{\ell}} v_j, \dots] = 0 \quad (j = 1, \dots, s)$$

where $\underline{\ell} = (\ell_1, \dots, \ell_4)$. F is a generalised function of the arguments shown.

Assuming that the highest order derivatives of v_j with respect to y_4 have order mj and that (without loss of generality) the highest order derivative in y_4 of all the v_j occurs in v_s. Then solving for this derivative (that is $\dfrac{\partial^{ms} v_j}{\partial y_4^{ms}}$), introduce the auxiliary dependent (state) variables x_i according to;

$$x_1 \triangleq v_1,\ x_2 \triangleq \frac{\partial v_1}{\partial y_4},\ \dots,\ x_{m1} \triangleq \frac{\partial^{m1-1} v_1}{\partial y_4^{m1-1}}$$

$$x_{m1+1} \triangleq v_2,\ x_{m1+2} \triangleq \frac{\partial v_2}{\partial y_4},\ \dots,\ x_{m1+m2} \triangleq \frac{\partial^{m2-1} v_2}{\partial y_4^{m2-1}}$$

. . . .

$$x_{\left[\sum_1^{s-1} mj\right]+1} \triangleq v_s, \qquad x_{\left[\sum_1^{s-1} mj\right]+2} \triangleq \frac{\partial v_s}{\partial y_4}, \ldots, \; x_{\left[\sum_1^{s} mj\right]} \triangleq \frac{\partial^{ms-1} v_s}{\partial y_4^{ms-1}}$$

and controls

$$u_1 \triangleq \frac{\partial^{m1} v_1}{\partial y_4^{m1}}, \; u_2 \triangleq \frac{\partial^{m2} v_2}{\partial y_4^{m2}}, \; \ldots, \; u_{s-1} \triangleq \frac{\partial^{m(s-1)} v_{(s-1)}}{\partial y_4^{m(s-1)}}$$

Differentiating the state variables $\{x_i;\ i = 1, \ldots, \sum_1^s mj = n\}$ with respect to y_4 yields the state equations of system type I - equation (c.3.1). Notice that derivatives with respect to y_4 on the left hand side are arbitrary and in fact any $y_1, \ldots, y_4$ could have been chosen; the reduction procedure remains the same. The highest derivatives of v_j with respect to y_4 in the above case for $j = 1, \ldots, (s-1)$ are chosen as control variables $\{u_k;\ k = 1, \ldots, (s-1) = r\}$. This satisfies the mathematical definition of control. The properties of the system will determine the choice of these variables. The reduction procedure extends readily to the case where (1-A1) are sets of simultaneous equations; the resulting state equations are suitably enlarged.

Notice that conventional-controls are interpreted as state variables. The controls in this case are the derivatives of the conventional-controls with respect to y_4. A general property of the decomposition algorithm is that the control occurs linearly in the resulting state equations. That is, this form of the decomposition is likely to lead to design cases which are formulated in a singular sense on attempting to apply the maximum principle. Hence to avoid singular formulations it will require a criterion nonlinear in the control and this often may not be the case. Notice the decomposition scheme of §C rarely, if ever, leads to state equations linear in the control, and hence the singularity of the formulation is independent in this case of the form of the criterion. (The final optimization solutions will in fact be the same no matter which decomposition scheme is employed. It is the awkwardness of the singularities which may occur in one case and which do not appeal.)

System type III: Consider the general partial differential equation

$$\text{(2-A1)} \qquad F[y_1, y_2, \ldots, v_j, \ldots, \partial_{\underline{\ell}} v_j, \ldots] = 0 \qquad (j = 1, \ldots, s)$$

defined over the y_1y_2-plane, where $\underline{\ell} = (\ell_1, \ell_2)$ and $\{v_j;\ j = 1, \ldots, s\}$ are equation dependent variables.

It is assumed that the highest order mixed derivatives of v_j with respect to y_1 and y_2 have order $2(mj)$ (clearly an even number) and that the highest order mixed derivative of all the v_j occurs in v_s and can be solved for. It is remarked that a system type III representation is inapplicable where the highest order (straight) derivative of any v_j is odd.

Introduce state variables according to

$$x_1 \triangleq v_1,\ x_2 \triangleq \frac{\partial^2 v_1}{\partial y_1 \partial y_2},\ \ldots,\ x_{m1} \triangleq \frac{\partial^{2m1-2} v_1}{\partial y_1^{m1-1} \partial y_2^{m1-1}}$$

$$x_{m1+1} \triangleq v_2,\ x_{m1+2} \triangleq \frac{\partial^2 v_2}{\partial y_1 \partial y_2},\ \ldots,\ x_{m1+m2} \triangleq \frac{\partial^{2m2-2} v_2}{\partial y_1^{m2-1} \partial y_2^{m2-1}}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x_{\left[\sum_1^{s-1} mj\right]+1} \triangleq v_s,\ x_{\left[\sum_1^{s-1} mj\right]+2} \triangleq \frac{\partial^2 v_s}{\partial y_1 \partial y_2},$$

$$\ldots\ x_{\left[\sum_1^{s} mj\right]} \triangleq \frac{\partial^{2ms-2} v_s}{\partial y_1^{ms-1} \partial y_2^{ms-1}}$$

and controls

$$u_1 \triangleq \frac{\partial^{2m1} v_1}{\partial y_1^{m1} \partial y_2^{m1}},\ u_2 \triangleq \frac{\partial^{2m2} v_2}{\partial y_1^{m2} \partial y_2^{m2}},\ \ldots,\ u_{(s-1)} \triangleq \frac{\partial^{2m(s-1)} v_{(s-1)}}{\partial y_1^{m(s-1)} \partial y_2^{m(s-1)}}$$

Taking the mixed y_1y_2 derivatives of the state gives a system of type III form - equation (C.3.3). Analogous comments apply to the above decomposition as were made for the decomposition to type I systems just considered.

APPENDIX TWO: STRUCTURAL RELIABILITY ANALYSIS

The reliability analysis problem is outlined with reference to the state space modelling adopted in the body of the thesis. Certain results relating to reliability detailed here are required for section §N on stochastic optimization under reliability constraints.

Comment on terminology: Structural reliability theory converses in the terms 'load' (or 'load effect') and 'resistance' as its basic building blocks. 'Load' is used to denote the response of the structural element and is therefore equivalent in intention to the concept of state in the present work. 'Resistance' is used to denote the 'capacity' of the structural element and has a one to one correspondence with the state. Terminology, apart from this interchange of usage of 'load' and state, follows essentially Freudenthal et al (1966) and Tichy and Vorlicek (1972).

General: The customary treatment of reliability has been restricted to structural elements whose behaviour may be described by a single entity. The concept of a state varying over the structure has always been regarded as too difficult. It became essential therefore to extend the level of usage of reliability ideas before reliability could be incorporated in the present work. (Tichy and Vorlicek (1972), it is noted, had previously attempted such a task by introducing the idea of a 'technical section' which is the part of the member in which failure occurs. For a member divided into several technical sections they generally assume that the resistances of the separate technical sections are mutually independent. Their intuitive approach will be seen to be contained within the following reliability analysis outline.)

Following Gnedenko et al (1969) (see also Tsypkin 1971), the three basic concepts of reliability theory may be stated as 'failure-free operation', 'life' and 'maintainability.' Consistent with the scope of the present work (in particular no learning, adaptive or changing systems) only the concept of failure-free operation is implied here in reference to reliability. 'Wear out failures', 'catastrophic failures' and reliability classified (in the sense of 'life') by time or the number of cycles are specifically excluded.

Assumptions: The variables of materials, geometry, loading and other external influences are random with known statistical characteristics; for example first, second and higher-order moments. The state will thus be random. Resistances however may be random or nonrandom depending on the problem at hand. The reliability formulation is given for random state, random resistance. A conversion to the special case of nonrandom properties may be easily accommodated by replacing the density functions with Dirac delta functions. An independence of state and resistance is assumed (Freudenthal 1963).

The structural system in general will be statically indeterminate. Markovian properties are taken to describe the state.

The reliability problem: The reliability analysis problem is solved in essentially three stages:

(a) The random state is determined everywhere throughout the structure and/or throughout time from given random geometries, loading and materials.

(b) At any value of the independent variable, the state is related to the resistance through their probability distributions to give a certain local probability of failure measure and reliability measure.

(c) The measure of reliability for the whole structure is chosen relating the local reliability measures.

When reliability is used in synthetic approaches as a constraint, only information relating to stages (b) and (c) are relevant; refer section §N. Stage (a) is the conventional structural analysis problem but with random variables.

Stage(a): For the equation

$$\text{(1-A2)} \qquad \underline{x}^{k+1} = \underline{\theta}[\underline{x}^k, \underline{p}^k] \qquad\qquad k = 0, 1, \ldots, N-1$$

defined at discrete points over the interval $[y^L, y^R]$, $\{\underline{p}^k = \underline{p}(k\Delta)$; $k = 0,1, \ldots, N-1\}$ and $\{\underline{x}^k = \underline{x}(k\Delta)$; $k = 0,1, \ldots, N\}$ are random sequences and take values at $y = k\Delta$. $\underline{x}^k$ is the n-dimensional state vector, $\underline{p}^k$ is the m-dimensional vector of random system parameters (and includes the control, which is given in the analysis problem, loading and others), and $\underline{\theta}$ is a real nonlinear n-vector function. The probability law of $\underline{p}^k$ is specified. Equation (1-A2) is a nonlinear stochastic vector difference equation; end conditions may be random or nonrandom. (The discretization scheme used to obtain (1-A2) from the continuous form is the same as that outlined in §E and §M. All processes in (1-A2) are random sequences and are completely defined by their 'finite dimensional distributions'.) The theory of stochastic difference equations is well delineated and their solution may be found in texts on stochastic processes (see for example Jazwinski 1970). The solution gives the sequence $\{\underline{x}^k;\ k = 0,1,\ \ldots\ , N\}$.

Stage (b): Having determined the state $\underline{x}(y)$ for all values of $y = k\Delta$ $k = 0,1,\ \ldots\ , N$ consider now the situation at any $k = i$, $0 \leq i \leq N$, i an integer.

The resistance of the structure at any i will be taken to be characterized by elements $\underline{r} \in R$. The permissible region of states corresponds to a region $S \subset R$ and the limit states are represented by the limit surface ∂S, the boundary of the permissible region. ∂S may be deterministic or stochastic. Failure of the system corresponds to the intersection of the state with this limit surface. Reliability may then be defined as the probability that the state remains within the permissible region S.

Formally, the probability of failure η^i, at $y = i\Delta$ may be defined as

$$\begin{aligned} \eta^i &= P\{\underline{x}^i \notin S\} \\ &= \int_{E^n} \int_{\underline{r}=0}^{\underline{x}} p(\underline{r})\,p(\underline{x})\,d\underline{r}\,d\underline{x} \end{aligned} \tag{2-A2}$$

Note that for many problems, it is only a particular component x_j of the state vector or several components of the vector which are of interest and not the complete vector. However the above provides the means of treating the general reliability problem in a single form, and emphasizes the underlying fundamental nature of a state decomposition of the system equations. Where reliability measures are not specified on all states, joint probability distributions and densities reduce to marginal distributions and densities. The special scalar version of (2-A2) is readily recognised as the usual form of probability of failure for structural elements described by a single entity. A direct relation is available between the vector and scalar cases as the joint density function is in fact a scalar quantity. For both positive and negative occurrences of state and resistance, the total probability of failure is the sum of the probabilities of failure resulting from positive states and negative states (see for example Freudenthal et al 1966).

Stage (c): Having determined the local probability of failure at $y = i\Delta$, a relationship is sought expressing the total probability of failure of the structure as a function of the local probabilities. For a local probability of failure η^i, $i = 0, 1, \ldots, N$ then the local probability of survival (that is, reliability) is $\mu^i = 1-\eta^i$.

As compared with statically determinate structures, where the survival of the whole requires the simultaneous survival of all elements for all time, failure of the general statically indeterminate structure assumes two forms:

(i) Local failure only varies the total probability of failure but does not imply complete structural failure (a 'parallel connection'). The ultimate limit state is an example. It implies a structure with a changing probability of failure for proportionally increasing load and is outside the scope of the present work.

(ii) Local failure implies complete structural failure (a 'series connection'). Serviceability limit states are examples for indeterminate structures while all failures in determinate structures are of this kind.

For statistically independent local reliabilities, the reliability of the whole, μ, is the intersection of these events

$$\mu = \prod_{i=0}^{N} \mu^i$$

$$= \prod_{i=0}^{N} (1-\eta^i)$$

and hence the total probability of failure of the structure is

$$\eta = 1 - \prod_{i=0}^{N} (1-\eta^i) \simeq \sum_{i=0}^{N} \eta^i \quad \text{for small } \eta^i \tag{3-A2}$$

where η^i is given by equation (2-A2).

APPENDIX THREE: A ONE-PARAMETER FAMILY OF SURFACES.

Section §H uses a one-parameter family of surfaces in a qualitative manner to derive Bellman's functional equation. This appendix shows that such a family can be constructed, for example, using a spherical polar coordinate system. The idea is due to R.S. Long and to the writer's knowledge, the details are unavailable elsewhere.

Consider a spherical coordinate system (ρ,θ,ϕ) (figure 1-A3,(a)).

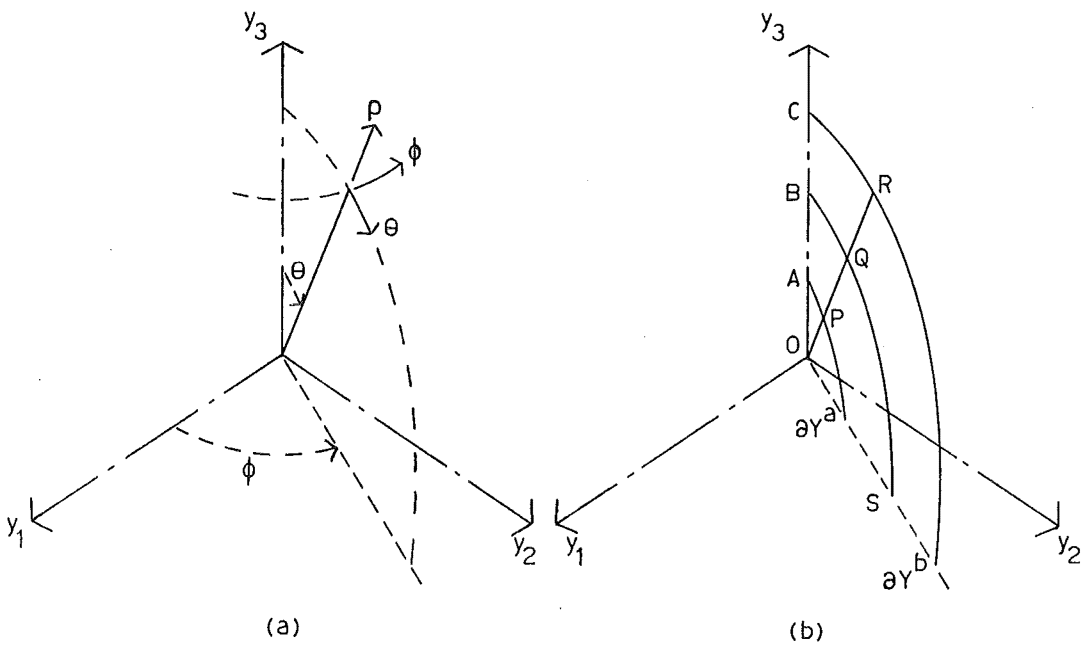

Figure 1-A3

Set $OP = \rho_1$, $OQ = \rho$, $OR = \rho_2$

$AC = \Gamma_0$, $AB = c\Gamma_0$

$PR = \Gamma$, $PQ = c\Gamma$

When $c = 0$, S coincides with ∂Y^a

$c = 1$, S coincides with ∂Y^b

Now $\rho = OQ = \rho_1 + c\Gamma = \rho_1 + c(\rho_2 - \rho_1)$

where ρ, ρ_1 and ρ_2 are all functions of θ,ϕ.

Then c defines a family of surfaces which moves from ∂Y^a to ∂Y^b as c goes from 0 to 1.

The transformation from the (y_1, y_2, y_3) coordinate system to the (ρ,θ,ϕ) system is

$$y_1 = \rho\sin\theta\cos\phi \qquad y_2 = \rho\sin\theta\sin\phi \qquad y_3 = \rho\cos\theta$$

for which the Jacobian $|J(\rho,\theta,\phi)| = \rho^2\sin\theta$.(See for example Sokolnikoff and Redheffer 1966.)